TM 9.56

- $\frac{dm}{dt} = kA = k4\pi r^2 \qquad (k>0)$
- $t=0$: $\left.\begin{array}{l} v(0)=v_0 \\ r(0)=r_0 \end{array}\right\}$ initial conditions

ⓐ Show that r increases linearly with t

- since droplet is spherical, it has volume $V = \frac{4}{3}\pi r^3$
  and mass $m = \rho V = \frac{4}{3}\pi r^3 \rho$ ⊛
- by assumption above, $\frac{dm}{dt} = k\pi r^2$
- explicitly differentiating our expression ⊛,

$$\frac{dm}{dt} = \frac{4}{3}\pi\rho\, 3r^2 \frac{dr}{dt} = 4\pi r^2 \frac{dr}{dt}\rho$$

- so:

$$k4\pi r^2 = 4\pi r^2 \rho \frac{dr}{dt} \longrightarrow \frac{dr}{dt} = \frac{k}{4\rho}$$

$$\boxed{r = \frac{k}{4\rho}t + r_0}$$

ⓑ Assume $r_0 \approx 0$ at $t=0$, and show $v(t) \sim t$

by Newton II, $F_{ext} = \frac{dp}{dt} = \frac{d}{dt}(mv)$

$$mg = \dot{m}v + m\dot{v}$$

$$\dot{v} = g - \frac{\dot{m}}{m}v$$

$$\dot{v} = g - \frac{4\pi r^2 \rho \dot{r}}{\frac{4}{3}\pi r^3 \rho} v$$

$$\dot{v} = g - \frac{3\dot{r}}{r} v$$

$$\dot{v} = g - \frac{3\alpha}{\alpha t + r_0} v \qquad \text{where } \alpha = \frac{k}{4\rho}$$

or

$$\dot{v} = g - \frac{3}{t + \frac{r_0}{\alpha}} v$$

This can be solved using Mathematica. The solution, upon taking $r_0 \to 0$, is $v(t) \approx \frac{gt}{4}$

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(* solve exact equation, including the initial condition v[0]=v0, then set r0→0 *)
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In[34]:= `DSolve[{v'[t] + 3/(t + r0/c) v[t] - g == 0, v[0] == v0}, v[t], t]`

Out[34]= $\left\{\left\{v[t] \to \frac{4\,g\,r0^3\,t + 6\,c\,g\,r0^2\,t^2 + 4\,c^2\,g\,r0\,t^3 + c^3\,g\,t^4 + 4\,r0^3\,v0}{4\,(r0 + c\,t)^3}\right\}\right\}$

In[35]:= solution = FullSimplify$\left[\frac{4\,g\,r0^3\,t + 6\,c\,g\,r0^2\,t^2 + 4\,c^2\,g\,r0\,t^3 + c^3\,g\,t^4 + 4\,r0^3\,v0}{4\,(r0 + c\,t)^3}\right]$

Out[35]= $\frac{g\,t\,(2\,r0 + c\,t)\,(2\,r0^2 + 2\,c\,r0\,t + c^2\,t^2) + 4\,r0^3\,v0}{4\,(r0 + c\,t)^3}$

In[36]:= `solution /. r0 → 0`

Out[36]= $\frac{g\,t}{4}$

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(* if you took r0→0 before solving the equation,
you get a result that's not so easy to deal with *)
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In[43]:= `DSolve[{v'[t] + 3/t v[t] - g == 0}, v[t], t]`

Out[43]= $\left\{\left\{v[t] \to \frac{g\,t}{4} + \frac{C[1]}{t^3}\right\}\right\}$

```
(* in this case, because of the t^-3 term,
we can't fix the constant C[1] using v[0]=0,
since the solution is divergent there.  so, this solution isn't really appropriate *)
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Easy to check by hand:

assuming $v(t) \approx At$, the eqn

$$\dot{v} = g - \frac{3}{t} v$$

becomes

$$A = g - 3A \implies 4A = g$$
$$A = \frac{g}{4}$$

so: $v = \frac{gt}{4}$ is consistent solution to the equation when $r_0 \to 0$.

(but with the caveat that it's not appropriate near $t=0$, or only with the assumption $V_0 \approx 0$)