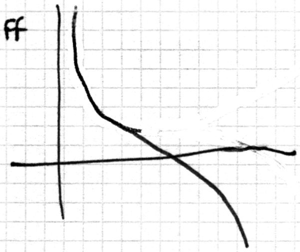
TM 8.8

$$U_{eff} = \frac{l^2}{2\mu r^2} - \frac{1}{2}kr^2$$



$F = kr \quad , \; k > 0$

$(U = -\frac{1}{2}kr^2) \quad \rightsquigarrow \quad F = -\frac{\partial U}{\partial r} = kr$

Starting from eq. 8.31 or 8.17:

$$\Theta = \int \frac{\frac{l}{r^2}\,dr}{\left[2\mu\left(E - U - \frac{l^2}{2\mu r^2}\right)\right]^{1/2}} + \text{const}$$

$$\Theta = \int \frac{(l/r^2)\,dr}{\sqrt{2\mu\left(E + \frac{1}{2}kr^2 - \frac{l^2}{2\mu r^2}\right)}} + \text{const}$$

$$= \frac{l}{\sqrt{2\mu}} \int \frac{dr}{r\sqrt{Er^2 + \frac{1}{2}kr^4 - \frac{l^2}{2\mu}}} + \text{const}$$

$$= \frac{l}{\sqrt{\mu k}} \int \frac{dr}{r\sqrt{r^4 + \frac{2E}{k}r^2 - \frac{l^2}{\mu k}}} + \text{const}$$

be a bit clever here:

$$\Theta = \frac{l}{\sqrt{\mu k}} \int \frac{r\,dr}{r^2\sqrt{r^4 + \frac{2E}{k}r^2 - \frac{l^2}{\mu k}}}$$

define $x = r^2$

$dx = 2r\,dr \longrightarrow r\,dr = \frac{dx}{2}$

$$= \frac{l}{2\sqrt{\mu k}} \int \frac{dx}{x\sqrt{x^2 + \frac{2E}{k}x - \frac{l^2}{\mu k}}}$$

Now use Eq E.10.b: with $a = 1$, $b = \frac{2E}{k}$, $c = -\frac{l^2}{\mu k}$ $\quad [b^2 > 0 > 4ac]$

$$\Theta + \text{const.} = \frac{l}{2\sqrt{\mu k}} \frac{1}{\sqrt{\frac{l^2}{\mu k}}} \sin^{-1}\left[\frac{bx + 2c}{|x|\sqrt{b^2 - 4ac}}\right]$$

$$= \frac{1}{2}\sin^{-1}\left[\frac{\frac{2E}{k}r^2 - 2\frac{l^2}{\mu k}}{r^2\sqrt{\left(\frac{2E}{k}\right)^2 + 4\frac{l^2}{\mu k}}}\right]$$

$$= \frac{1}{2}\sin^{-1}\left[\frac{1 - \frac{1}{r^2}\left(\frac{l^2}{k}\right)\left(\frac{k}{E}\right)\frac{1}{\mu}}{\sqrt{1 + \left(\frac{l^2}{k}\right)\left(\frac{k}{E}\right)^2\frac{1}{\mu}}}\right]$$

$$\Theta + \text{const.} = \frac{1}{2}\sin^{-1}\left[\frac{1 - \frac{1}{r^2}\frac{l^2}{\mu E}}{\sqrt{1 + \frac{l^2 k}{\mu E^2}}}\right]$$

define $\epsilon = \sqrt{1 + \frac{l^2 k}{\mu E^2}}$

$\alpha = \frac{l^2}{\mu E}$

similar to 8.40.

TM 8.8

$$2(\theta + \text{const}) = \sin^{-1}\left[\frac{1-\frac{\alpha}{r^2}}{\epsilon}\right]$$

$$\sin[2(\theta + \text{const})] = \frac{1}{\epsilon}\left(1 - \frac{\alpha}{r^2}\right)$$

- if the const. is chosen $\frac{\pi}{4}$, we have $\sin[2\theta + \frac{\pi}{2}] = -\cos 2\theta$

so: $-\cos 2\theta = \frac{1}{\epsilon}\left(1 - \frac{\alpha}{r^2}\right)$

$$-\epsilon \cos 2\theta = 1 - \frac{\alpha}{r^2}$$

$$\frac{\alpha}{r^2} = 1 + \epsilon \cos 2\theta$$ which looks quite a bit like 8.41

Is it a hyperbola?

$$x = r\cos\theta$$
$$y = r\sin\theta$$

our eqn is: $\alpha = r^2(1 + \epsilon \cos 2\theta)$

$$\alpha = r^2(1 + \epsilon(\cos^2\theta - \sin^2\theta))$$
$$= r^2 + \epsilon(r\cos\theta)^2 - \epsilon(r\sin\theta)^2$$
$$\alpha = x^2 + y^2 + \epsilon x^2 - \epsilon y^2$$
$$\alpha = x^2(1+\epsilon) + y^2(1-\epsilon) \qquad (\epsilon > 1)$$

$$1 = \frac{x^2}{\left(\frac{\alpha}{1+\epsilon}\right)} - \frac{y^2}{\left(\frac{\alpha}{\epsilon - 1}\right)}$$

$$1 = \left(\frac{x}{\sqrt{\frac{\alpha}{\epsilon+1}}}\right)^2 - \left(\frac{y}{\sqrt{\frac{\alpha}{\epsilon-1}}}\right)^2$$

which is indeed the form of a hyperbola.