$F = \frac{-k}{r^2} e^{-r/a}$; Are circular orbits stable?

- circular orbits are stable if, for r_{min} defined by $\left.\frac{\partial U_{eff}}{\partial r}\right|_{r_{min}} = 0$

we have:

$$\frac{\partial^2 U_{eff}}{\partial r^2_{min}} > 0.$$

- $U_{eff} = \frac{l^2}{2\mu r^2} + U$

$$\frac{\partial U_{eff}}{\partial r} = -\frac{l^2}{\mu r^3} + \frac{\partial U}{\partial r} = -\frac{l^2}{\mu r^3} - F = -\frac{l^2}{\mu r^3} + \frac{k}{r^2} e^{-r/a}$$

since $F = -\frac{\partial U}{\partial r}$

$$\left.\frac{\partial U_{eff}}{\partial r}\right|_{r_{min}} = 0 \implies 0 = \frac{-l^2}{\mu r^3_{min}} + \frac{k}{r^2_{min}} e^{-r_{min}/a}$$

$$\frac{l^2}{\mu r_m} = k e^{-r_m/a}$$

or $r_m = \frac{l^2}{\mu k} e^{r_m/a}$

This is a transcendental eqn. that we can't solve explicitly.

- $$\frac{\partial^2 U_{eff}}{\partial r^2} = \frac{\partial}{\partial r}\left[-\frac{l^2}{\mu r^3} + \frac{k}{r^2} e^{-r/a}\right]$$

$$= +\frac{3l^2}{\mu r^4} - \frac{2k}{r^3} e^{-r/a} + \frac{k}{r^2}\left(-\frac{1}{a}\right) e^{-r/a}$$

$$= \frac{1}{r^2}\left[\frac{3l^2}{\mu r^2} + k e^{-r/a}\left(-\frac{2}{r} - \frac{1}{a}\right)\right]$$

$$= \frac{1}{r^2}\left[\frac{3l^2}{\mu r^2} - k e^{-r/a}\left(\frac{2}{r} + \frac{1}{a}\right)\right]$$

at $r = r_{min}$,

$$\left.\frac{\partial^2 U_{eff}}{\partial r^2}\right|_{r_m} = \frac{1}{r_m^2}\left[\frac{3l^2}{\mu r_m^2} - k e^{-r_m/a}\left(\frac{2}{r_m} + \frac{1}{a}\right)\right]$$

use $e^{-r_m/a} = \frac{l^2}{\mu r_m k}$ $\implies$ $$\left.\frac{\partial^2 U_{eff}}{\partial r^2}\right|_{r_m} = \frac{1}{r_m^2}\left[\frac{3l^2}{\mu r_m^2} - k\frac{l^2}{\mu k r_m}\left(\frac{2}{r_m} + \frac{1}{a}\right)\right]$$

$$= \frac{1}{r_m^2}\left[\frac{3l^2}{\mu r_m^2} - \frac{l^2}{\mu r_m}\left(\frac{2}{r_m} + \frac{1}{a}\right)\right]$$

$$= \frac{1}{r_m^2}\left[\frac{l^2}{\mu r_m^2} - \frac{l^2}{\mu r_m a}\right]$$

$$\left.\frac{\partial^2 U_{eff}}{\partial r^2}\right|_{r_m} = \frac{1}{r_m^2}\frac{\ell^2}{\mu r_m^2}\left(1-\frac{r_m}{a}\right) = \frac{\ell^2}{\mu r_m^4}\left(1-\frac{r_m}{a}\right)$$

so: circular orbits are stable if $r_m < a$.

Can we say more? go back to the $\left.\frac{\partial U_{eff}}{\partial r}\right|_{r_m} = 0$ condition...

rewrite as:

$$\frac{r_m}{a} e^{-r_m/a} = \frac{\ell^2}{\mu k a}$$

$$x e^{-x} = c \quad , \begin{cases} c = \frac{\ell^2}{\mu k a} \\ x = \frac{r_m}{a} \end{cases}$$

If we look at $x e^{-x}$ in Mathematica, we see the fn has a peak at $x=1$ (check: $\frac{d}{dx}(xe^{-x}) = e^{-x} - xe^{-x}$
$= (1-x)e^{-x}$
$= 0$ at $x=1$)

At $x=1$, $e^{-x}x = e^{-1}$. For each $c < \frac{1}{e}$ there are two solutions to $xe^{-x} = c$, one with $x>1$ and one with $x<1$. The $x>1$ solution is then an UNSTABLE circular orbit, the $x<1$ solution is a STABLE circular orbit.

so:
- for $\frac{\ell^2}{\mu k a} < \frac{1}{e}$, there is a stable circular orbit at $r<a$, and an unstable one at $r>a$.
- for $\frac{\ell^2}{\mu k a} > \frac{1}{e}$ there are no circular orbits. ($\left.\frac{\partial U}{\partial r}\right|_{r_m} = 0$ has not solutions)

→ large ℓ means strong centrifugal force, which will eventually overwhelm the attractive force for sufficiently large ℓ.

→ a is the 'cutoff' range of the attractive force. If a is very small it cuts off too quickly to stabilize the centrifugal force. If a is very large, the force looks like the Kepler force

→ k is the strength of the kepler force strength.

TM 8.32


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In[2]:= Plot[x Exp[-x], {x, 0, 5}]
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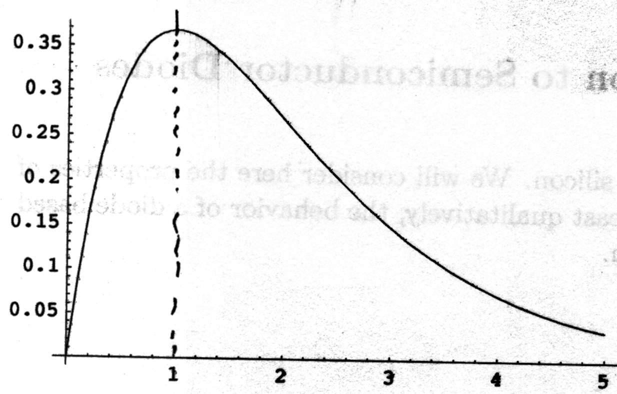


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Out[2]= - Graphics -
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