

$$F = -\frac{k}{r^2} e^{-r/a} \quad \text{Are circular orbits stable?}$$

- circular orbits are stable if, for  $r_{\min}$  defined by  $\frac{\partial U_{\text{eff}}}{\partial r} \Big|_{r_{\min}} = 0$  we have:

$$\frac{\partial^2 U_{\text{eff}}}{\partial r^2} \Big|_{r_{\min}} > 0.$$

$$U_{\text{eff}} = \frac{l^2}{2\mu r^2} + U$$

$$\frac{\partial U_{\text{eff}}}{\partial r} = -\frac{l^2}{\mu r^3} + \frac{\partial U}{\partial r} = -\frac{l^2}{\mu r^3} - F = -\frac{l^2}{\mu r^3} + \frac{k}{r^2} e^{-r/a}$$

$$\text{since } F = -\frac{\partial U}{\partial r}$$

$$\frac{\partial U_{\text{eff}}}{\partial r} \Big|_{r_{\min}} = 0 \implies 0 = -\frac{l^2}{\mu r_{\min}^3} + \frac{k}{r_{\min}^2} e^{-r_{\min}/a}$$

$$\frac{l^2}{\mu r_{\min}} = k e^{-r_{\min}/a}$$

$$\text{or } r_{\min} = \frac{l^2}{\mu k} e^{r_{\min}/a}$$

This is a transcendental eqn. that we can't solve explicitly.

$$\begin{aligned} \frac{\partial^2 U_{\text{eff}}}{\partial r^2} &= \frac{\partial}{\partial r} \left[ -\frac{l^2}{\mu r^3} + \frac{k}{r^2} e^{-r/a} \right] \\ &= +\frac{3l^2}{\mu r^4} - \frac{2k}{r^3} e^{-r/a} + \frac{k}{r^2} \left( -\frac{1}{a} \right) e^{-r/a} \\ &= \frac{1}{r^2} \left[ \frac{3l^2}{\mu r^2} + k e^{-r/a} \left( -\frac{2}{r} - \frac{1}{a} \right) \right] \\ &= \frac{1}{r^2} \left[ \frac{3l^2}{\mu r^2} - k e^{-r/a} \left( \frac{2}{r} + \frac{1}{a} \right) \right] \end{aligned}$$

at  $r = r_{\min}$ ,

$$\frac{\partial^2 U_{\text{eff}}}{\partial r^2} \Big|_{r_{\min}} = \frac{1}{r_{\min}^2} \left[ \frac{3l^2}{\mu r_{\min}^2} - k e^{-r_{\min}/a} \left( \frac{2}{r_{\min}} + \frac{1}{a} \right) \right]$$

$$\text{use } e^{-r_{\min}/a} = \frac{l^2}{\mu r_{\min} k} \implies$$

$$\begin{aligned} \frac{\partial^2 U_{\text{eff}}}{\partial r^2} \Big|_{r_{\min}} &= \frac{1}{r_{\min}^2} \left[ \frac{3l^2}{\mu r_{\min}^2} - k \frac{l^2}{\mu k r_{\min}} \left( \frac{2}{r_{\min}} + \frac{1}{a} \right) \right] \\ &= \frac{1}{r_{\min}^2} \left[ \frac{3l^2}{\mu r_{\min}^2} - \frac{l^2}{\mu r_{\min}} \left( \frac{2}{r_{\min}} + \frac{1}{a} \right) \right] \\ &= \frac{1}{r_{\min}^2} \left[ \frac{l^2}{\mu r_{\min}^2} - \frac{l^2}{\mu r_{\min} a} \right] \end{aligned}$$

$$\left. \frac{\partial^2 U_{\text{eff}}}{\partial r^2} \right|_{r_m} = \frac{1}{r_m^2} \frac{l^2}{\mu r_m^2} \left( 1 - \frac{r_m}{a} \right) = \frac{l^2}{\mu r_m^4} \left( 1 - \frac{r_m}{a} \right)$$

(2)

So: circular orbits are stable if  $r_m < a$ .

Can we say more? go back to the  $\left. \frac{\partial U_{\text{eff}}}{\partial r} \right|_{r_m} = 0$  condition...

rewrite as:

$$\frac{r_m}{a} e^{-r_m/a} = \frac{l^2}{\mu k a}$$

$$x e^{-x} = c \quad , \quad \begin{cases} c = \frac{l^2}{\mu k a} \\ x = \frac{r_m}{a} \end{cases}$$

If we look at  $x e^{-x}$  in mathematics, we see the fn has a peak at  $x=1$  (check:  $\frac{d}{dx}(x e^{-x}) = e^{-x} - x e^{-x} = (1-x)e^{-x} = 0$  at  $x=1$ )

At  $x=1$ ,  $e^{-x} x = e^{-1}$ . For each  $c < \frac{1}{e}$  there are two solutions to  $x e^{-x} = c$ , one with  $x > 1$  and one with  $x < 1$ . The  $x > 1$  solution is thus an UNSTABLE circular orbit, the  $x < 1$  solution is a STABLE circular orbit.

So: for  $\frac{l^2}{\mu k a} < \frac{1}{e}$ , there is a stable circular orbit at  $r < a$ , and an unstable one at  $r > a$ .

• for  $\frac{l^2}{\mu k a} > \frac{1}{e}$  there are no circular orbits.  
( $\left. \frac{\partial U}{\partial r} \right|_{r_m} = 0$  has no solutions)

→ large  $l$  means strong centrifugal force, which will eventually overwhelm the attractive force for sufficiently large  $l$ .

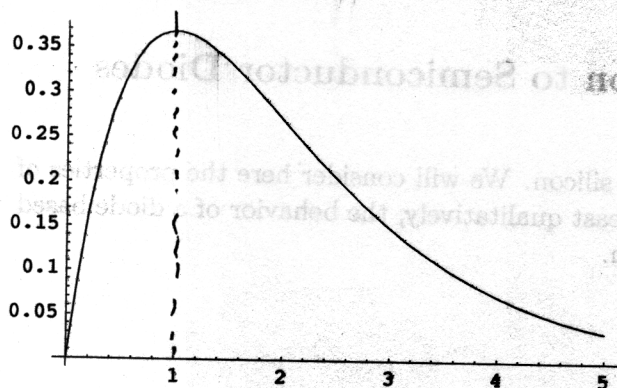
→  $a$  is the 'cutoff' range of the attractive force. If  $a$  is very small it cuts off too quickly to stabilize the centrifugal force. If  $a$  is very large, the force looks like the Kepler force.

→  $k$  is the strength of the Kepler force strength.



TM 8.32

In[2]:= Plot[x Exp[-x], {x, 0, 5}]



Out[2]= - Graphics -

Physics 17 - Lab 4 - Winter-Spring 2005

## 4.3 Supplement: Introduction to Semiconductor Diodes

Many semiconductor diodes are made from silicon. We will consider here the properties of silicon to show how we can understand, at least qualitatively, the behavior of a diode on the basis of electrical charges in silicon.

### 4.3.1 Pure Semiconductors

A silicon nucleus contains 14 protons, giving it a charge of  $+14$  units. A neutral Si atom also has 14 electrons, of which ten are tightly bound to the nucleus. We can therefore think of a Si atom as having a central core (charge  $= +4$ ), surrounded by four rather loosely attached electrons (valence electrons) — represented by the black dots (Fig. 4.11).

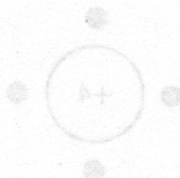


Figure 4.11

A piece of pure Si is just a collection of these atoms stuck together in a crystal (Fig. 4.12).

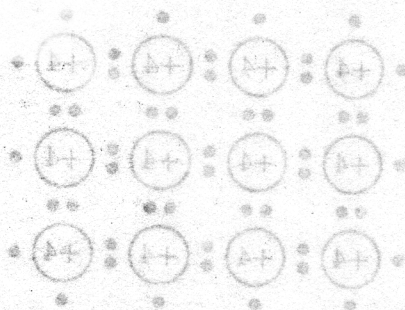


Figure 4.12

The outer (or "valence") electrons prefer to be distributed four to each Si. As you can see from the diagram, neighboring atoms share valence electrons as they form chemical bonds, and these bonds hold the solid together.

Pure Si is a pretty good insulator at low temperatures, i.e., near absolute zero. But at slightly higher temperatures Si is not a perfect insulator, since thermal agitation knocks some of the valence electrons loose from their atoms, and separation of an electron can make the atom a free electron.