TM 8.22 : [see ~~eqns~~ ~~8.1~~ examples 8.1 and 8.2 also]

$$F = -\frac{k}{r^3} \quad (k>0) \quad \longrightarrow \quad U = -\frac{k}{2r^2}$$

• Starting from eq. 8.20

$$U_{eff} = \left(\frac{\ell^2}{2\mu} - \frac{k}{2}\right)\frac{1}{r^2} = \frac{\ell^2}{2\mu r^2}\left(1 - \frac{\mu k}{\ell^2}\right)$$

$$\frac{d^2u}{d\theta^2} + u = -\frac{\mu}{\ell^2}\frac{1}{u^2}F$$

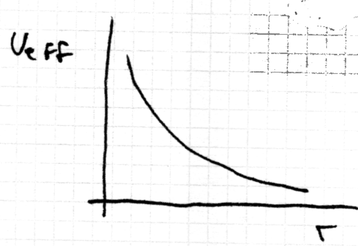
where $u = \frac{1}{r}$

• here, $F = -ku^3$, so we have

$$u'' + u = +\frac{\mu k u}{\ell^2}$$

$$u'' + \left(1 - \frac{\mu k}{\ell^2}\right)u = 0$$

CASE 1 : $1 - \frac{\mu k}{\ell^2} \equiv \alpha > 0$. The u equation is that of a harmonic oscillator



$$u'' + \alpha u = 0$$

which has solution

$$u = A\cos\left(\sqrt{\alpha}(\theta - \delta)\right)$$

where A and δ are set by initial conditions

we can choose our coord. system so that $\delta = 0$. Then

$$u = \frac{1}{r} = A\cos(\sqrt{\alpha}\,\theta).$$

$$r = \frac{1}{A\cos(\sqrt{\alpha}\,\theta)}$$

If $\theta = 0$ is one of the points in our orbit, then A must be positive (so that $r = \frac{1}{A} > 0$)

The denominator reaches a maximum at $\theta = 0$, so this is $r_{min} = \frac{1}{A}$. The denominator vanishes at

$$\sqrt{\alpha}\,\theta = \pm\frac{\pi}{2}$$

$$\theta = \pm\frac{1}{\sqrt{\alpha}}\frac{\pi}{2}.$$ At these points, $r = \infty$

so : particle approaches from $r = \infty$, makes a closest approach at $\theta = 0$, then flies off back to ∞. (This is the analog of the hyperbolic orbit in Kepler problem). It spans an angle of $\frac{\pi}{\sqrt{\alpha}}$ in total, so if $\sqrt{\alpha} \leq \frac{1}{2}$ it makes a full loop about the force center.

[That this corresponds to a positive energy solution is clear from the trajectory (it goes to and from infinity), but it would be nice to find the energy explicitly, or equivalently to express A in terms of the energy.

To do this, we use

$$E = T + U$$
$$= \frac{1}{2}\mu(\dot r^2 + r^2\dot\theta^2) - \frac{k}{2}u^2$$

From p. 292, $\quad \dot r = -\frac{\ell}{\mu}u'$

$$\dot\theta = \frac{\ell}{\mu r^2} = \frac{\ell}{\mu}u^2 \quad \Rightarrow r^{\cancel{2}}\dot\theta = \frac{\ell}{\mu}u$$

so: $\quad \dot r^2 + (r\dot\theta)^2 = \left(\frac{\ell}{\mu}\right)^2(u'^2 + u^2)$

and $\quad E = \frac{1}{2}\mu\left[\left(\frac{\ell}{\mu}\right)^2(u'^2+u^2)\right] - \frac{k}{2}u^2$

$$= \frac{1}{2}\frac{\ell^2}{\mu}u'^2 + \frac{1}{2}\left(\frac{\ell^2}{\mu} - k\right)u^2$$
$$= \frac{1}{2}\frac{\ell^2}{\mu}u'^2 + \frac{1}{2}\frac{\ell^2}{\mu}\left(1 - \frac{k\mu}{\ell^2}\right)u^2$$
$$= \frac{1}{2}\frac{\ell^2}{\mu}\left[u'^2 + \left(1 - \frac{k\mu}{\ell^2}\right)u^2\right]$$

In <u>case I</u>,

$$u = A\cos(\sqrt{\alpha}\,\theta) \longrightarrow u' = -A\sqrt{\alpha}\,\sin(\sqrt{\alpha}\,\theta)$$

and $\quad E = \frac{1}{2}\frac{\ell^2}{\mu}[u'^2 + \alpha u^2]$

$$= \frac{1}{2}\frac{\ell^2}{\mu}\left[A^2\alpha\sin^2(\sqrt{\alpha}\,\theta) + A^2\alpha\cos^2(\sqrt{\alpha}\,\theta)\right]$$

$$E = \frac{1}{2}\frac{\ell^2}{\mu}A^2\alpha \quad > 0$$

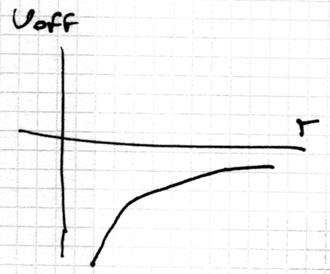
so : $\quad A = \sqrt{\dfrac{2\mu E}{\alpha}}$

$$r = \frac{1}{\cos(\sqrt{\alpha}\,\theta)}\sqrt{\frac{\alpha}{2\mu E}}$$

CASE II $\quad 1 - \frac{\mu k}{\ell^2} \equiv -|\alpha| < 0$

The solution to $u'' - |\alpha| u = 0$ is a pair of pure exponentials, ③

$$u = c_1 e^{-\sqrt{|\alpha|}\,\theta} + c_2 e^{\sqrt{|\alpha|}\,\theta}.$$



This is equivalent to a pair of hyperbolic trig functions (see Appendix D to translate from exponentials to sinh and cosh), and results are a bit prettier in this form, so I'll write

$$u = A\cosh(\sqrt{|\alpha|}\,\theta) + B\sinh(\sqrt{|\alpha|}\,\theta)$$

$$\cosh x = \frac{e^x + e^{-x}}{2} \quad (D.46)$$

$$\sinh x = \frac{e^x - e^{-x}}{2} \quad (D.45)$$

If u_0 is the value of u at $\theta = 0$, we can write

$$u_0 = A\,\underbrace{\cosh(0)}_{1} + B\,\underbrace{\sinh(0)}_{0} = A$$

So:

$$u(\theta) = u_0 \cosh(\sqrt{|\alpha|}\,\theta) + B\sinh(\sqrt{|\alpha|}\,\theta)$$

or:

$$\frac{1}{r} = \frac{1}{r_0}\cosh(\sqrt{|\alpha|}\,\theta) + B\sinh(\sqrt{|\alpha|}\,\theta)$$

- For $r_0 = \infty$, $\frac{1}{r_0} = 0$ and $\frac{1}{r} = B\sinh(\sqrt{|\alpha|}\,\theta)$. (B must be >0)

So: $r = \dfrac{1}{B\sinh(\sqrt{|\alpha|}\,\theta)}$ particle

· for θ large (positive), $r \sim \frac{2}{B} e^{-\sqrt{|\alpha|}\,\theta}$ i.e. particle falls to center, spiralling infinitely many times.

- For r_0 not infinite,

$$\frac{1}{r} = \frac{1}{r_0}\cosh(\sqrt{|\alpha|}\,\theta) + B\sinh(\sqrt{|\alpha|}\,\theta)$$

$$= \frac{1}{r_0}\cosh(\sqrt{|\alpha|}\,\theta)\left(1 + \underbrace{Br_0}_{\text{call it } b}\tanh(\sqrt{|\alpha|}\,\theta)\right)$$

$$\frac{1}{r} = \frac{1}{r_0}\cosh(\sqrt{|\alpha|}\,\theta)\left(1 + b\tanh\sqrt{|\alpha|}\,\theta\right)$$

$\tanh x$ runs from -1 at $x = -\infty$ to 1 at $x = \infty$

so for $|b| < 1$, $\frac{1}{r}$ is always positive for all θ.

In that case, for $\theta \to \infty$

$$\frac{1}{r} \sim \frac{1+b}{r_0}\frac{e^{\sqrt{|\alpha|}\,\theta}}{2}, \text{ or } r \sim \frac{r_0}{2(1+b)} e^{-\sqrt{|\alpha|}\,\theta}$$

i.e. spiral to center

and for $\theta \to -\infty$

$$r \sim \frac{r_0}{2(1-b)} e^{-\sqrt{|\alpha|}\,|\theta|}$$

also spiral to center. So, this is a bounded orbit. Must correspond to $E<0$.

(Case II) For $|b| > 1$ $r \to \infty$ for $1 + b \tanh(\sqrt{|\alpha|}\,\theta) = 0$

$$\tanh(\sqrt{|\alpha|}\,\theta) = -\frac{1}{b}$$

$$\theta_\infty = \frac{\operatorname{Arctanh}\left(-\frac{1}{b}\right)}{\sqrt{|\alpha|}} < 0$$

i.e. particle comes in from infinity at that angle.

<u>But</u>, then we could have taken θ_∞ to be the $\theta = 0$ line, and this is just the $r_0 = \infty$ case again. Still spirals to center!

Case II : relating parameters to energy

$$E = \frac{1}{2}\frac{\ell^2}{\mu}\left[u'^2 + \left(1 - \frac{k\mu}{\ell^2}\right)u^2\right]$$

$$= \frac{1}{2}\frac{\ell^2}{\mu}\left[u'^2 - |\alpha| u^2\right]$$

$$u = A\cosh(\sqrt{|\alpha|}\,\theta) + B\sinh(\sqrt{|\alpha|}\,\theta)$$

$$u' = \sqrt{|\alpha|}\left(A\sinh(\sqrt{|\alpha|}\,\theta) + B\cosh(\sqrt{|\alpha|}\,\theta)\right)$$

so:

$$u'^2 - |\alpha| u^2 =$$

$$|\alpha|\left(\left(A\sinh\sqrt{|\alpha|}\,\theta + B\cosh(\sqrt{|\alpha|}\,\theta)\right)^2 - \left(A\cosh(\sqrt{|\alpha|}\,\theta) + B\sinh\sqrt{|\alpha|}\,\theta\right)^2\right) =$$

$$|\alpha|\left(A^2(\sinh^2 - \cosh^2) + B(\cosh^2 - \sinh^2)\right) =$$

$$|\alpha|\,(B^2 - A^2)$$

Thus

$$E = \frac{1}{2}\frac{\ell^2}{\mu}|\alpha|(B^2 - A^2) = \frac{1}{2}\frac{\ell^2}{\mu}|\alpha|\left(B^2 - \frac{1}{r_0^2}\right) = \frac{1}{2}\frac{\ell^2}{\mu r_0^2}|\alpha|(b^2 - 1)$$

For $b^2 < 1$, $E < 0$

For $b^2 > 1$, $E > 0$

$$\text{or}: \quad b^2 = \frac{2E\mu r_0^2}{\ell^2}\frac{1}{|\alpha|} + 1$$

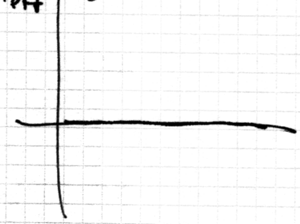
$$b = \pm\sqrt{1 + \frac{2\mu_0 r_0^2 E}{\ell^2 |\alpha|}}$$

$$\text{or} \quad B = \pm\sqrt{\frac{1}{r_0^2} + \frac{2\mu_0 E}{\ell^2|\alpha|}}$$

CASE III : $1 - \frac{\mu k}{\ell^2} = 0 \longrightarrow u'' = 0$

$$u = A(\theta - \delta) = \frac{1}{r}$$

$U_{eff} = 0$



$$r = \frac{1}{A(\theta - \delta)}$$

orient our axes so that $\delta = 0$. Then $r = \frac{1}{A\theta}$.

- At $\theta = 0$, $r = \infty$, so particle comes in from ∞.
- For $\theta \to \infty$, $r \to 0$. So, particle spirals to center.

What's the energy?

$$E = \frac{1}{2}\frac{\ell^2}{\mu}\left[u'^2 + \cancel{u^2}/\alpha\right] = \frac{1}{2}\frac{\ell^2}{\mu}u'^2 \geq 0$$

$$= \frac{1}{2}\frac{\ell^2}{\mu}A^2 \longrightarrow A = \sqrt{\frac{2E\mu}{\ell^2}}$$

and so $r = \sqrt{\frac{\ell^2}{2E\mu}}\,\frac{1}{\theta}$ $(\theta > 0)$

Are circular orbits stable?

No. A circular orbit requires $u = \text{const}$, $u' = u'' = 0$, which is not one of the allowed solutions

Another way to see it:

circular orbit is stable if

$$\left.\frac{\partial U_{eff}}{\partial r}\right|_{r_{min}} = 0$$

$$\left.\frac{\partial^2 U_{eff}}{\partial r^2}\right|_{r_{min}} > 0.$$

In this case, $\left.\frac{\partial U_{eff}}{\partial r}\right|_{r_{min}} = 0$

$$-\frac{\ell^2}{\mu r_{min}^3}\left(1 - \frac{\mu k}{\ell^2}\right) = 0 \Rightarrow \frac{\mu k}{\ell^2} = 1$$

But then

$U_{eff} = 0$!

And we've seen (case III) that this doesn't give circular orbits