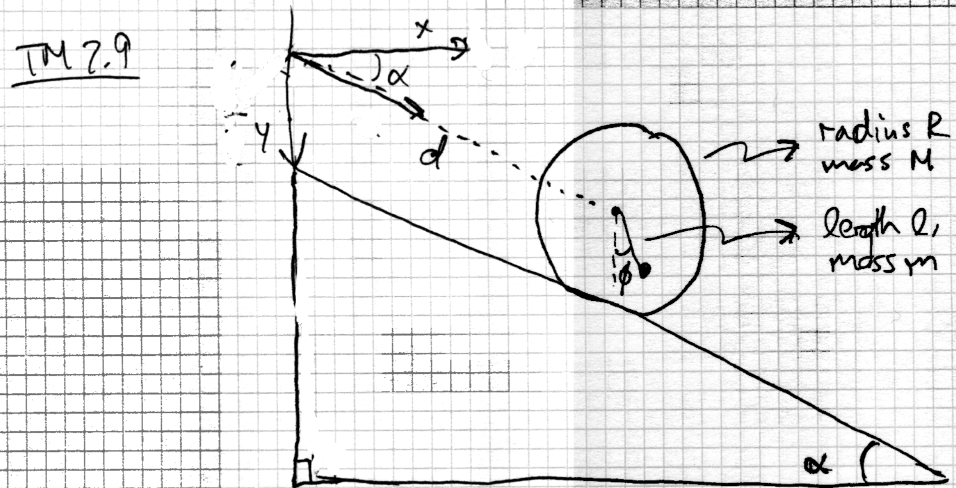
TM 7.9



$T_{disk} = \frac{1}{2} M v_{cm}^2 + \frac{1}{2} I \omega^2$, where I is moment of inertia $= MR^2$, ω is angular speed about CM

rolling without slipping criterion : $v_{cm} = R\omega$

so

$$T_{disk} = \frac{1}{2} M v_{cm}^2 + \frac{1}{2} MR^2 \left(\frac{v_{cm}}{R}\right)^2$$
$$= M v_{cm}^2$$
$$= M \dot{d}^2$$

$x_{cm} = d\cos\alpha$

$y_{cm} = d\sin\alpha$

$$U_{disk} = -Mgy$$
$$= -Mgd\sin\alpha$$

$$T_{pendulum} = \frac{1}{2} m v_p^2 = \frac{1}{2} m(\dot{x}_p^2 + \dot{y}_p^2)$$
$$= \frac{1}{2} m\left[(\dot{d}\cos\alpha + l\cos\phi\dot{\phi})^2 + (\dot{d}\sin\alpha - l\sin\phi\dot{\phi})^2\right]$$
$$= \frac{1}{2} m\left[\dot{d}^2(\cos^2\alpha + \sin^2\alpha) + (l\dot{\phi})^2(\sin^2\phi + \cos^2\phi) + 2\dot{d}l\dot{\phi}\cos\alpha\cos\phi - 2\dot{d}l\dot{\phi}\sin\alpha\sin\phi\right]$$
$$= \frac{1}{2} m\left[\dot{d}^2 + l^2\dot{\phi}^2 + 2\dot{d}l\dot{\phi}\,(\underbrace{\cos\alpha\cos\phi - \sin\alpha\sin\phi}_{\cos(\alpha+\phi)}\right]$$
$$= \frac{1}{2} m\left[\dot{d}^2 + l^2\dot{\phi}^2 + 2\dot{d}l\dot{\phi}\cos(\alpha+\phi)\right]$$

$x_p = x_{cm} + l\sin\phi$
$= d\cos\alpha + l\sin\phi$

$y_p = y_{cm} + l\cos\phi$
$= d\sin\alpha + l\cos\phi$

$$U_{pend} = -mgy_p = -mg(d\sin\alpha + l\sin\phi)$$

$$L = T_{disk} + T_{pendulum} - (U_{disk} + U_{pendulum})$$
$$= M\dot{d}^2 + \frac{1}{2} m\left[\dot{d}^2 + l^2\dot{\phi}^2 + 2l\dot{d}\dot{\phi}\cos(\alpha+\phi)\right]$$
$$+ Mgd\sin\alpha + mg(d\sin\alpha + l\sin\phi)$$

Lagrange's eqns:

for d: $$\frac{\partial L}{\partial d} - \frac{d}{dt}\frac{\partial L}{\partial \dot d} = 0$$

$$\frac{\partial L}{\partial d} = (M+m) g \sin\alpha$$

$$\frac{\partial L}{\partial \dot d} = 2\left(M+\frac{m}{2}\right)\dot d + ml\dot\phi\cos(\alpha+\phi)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot d} = (2M+m)\ddot d + ml\ddot\phi\cos(\alpha+\phi) - ml\dot\phi^2\sin(\alpha+\phi)$$

so:

$$\boxed{(2M+m)\ddot d + ml\ddot\phi\cos(\alpha+\phi) - ml\dot\phi^2\sin(\alpha+\phi) = (M+m) g \sin\alpha}$$

for ϕ:

$$\frac{\partial L}{\partial \phi} - \frac{d}{dt}\frac{\partial L}{\partial \dot\phi} = 0$$

$$\frac{\partial L}{\partial \phi} = -ml\dot d\dot\phi\sin(\alpha+\phi) + mgl\cos\phi$$

$$\frac{\partial L}{\partial \dot\phi} = ml^2\dot\phi + ml\dot d\cos(\alpha+\phi)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot\phi} = ml^2\ddot\phi + ml\ddot d\cos(\alpha+\phi) - ml\dot d\dot\phi\sin(\alpha+\phi)$$

so:

$$ml^2\ddot\phi + ml\ddot d\cos(\alpha+\phi) - ml\dot d\dot\phi\sin(\alpha+\phi) = -ml\dot d\dot\phi\sin(\alpha+\phi) + mgl\cos\phi$$

$$l\ddot\phi + \ddot d\cos(\alpha+\phi) - \cancel{\dot d\dot\phi\sin(\alpha+\phi)} = -\cancel{\dot d\dot\phi\sin(\alpha+\phi)} + g\cos\phi$$

$$\boxed{\ddot\phi + \frac{\ddot d}{l}\cos(\alpha+\phi) - \frac{g}{l}\cos\phi = 0}$$