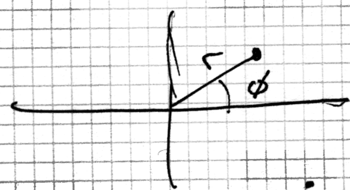
TM 7.4

$f = -Ar^{\alpha-1}$ A and α (>0) are constants

$U(r=0) = 0$

Choose polar coordinates



$$T = \tfrac{1}{2}mv^2$$
$$= \tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2)$$
$$U = -Ar^{\alpha-1}$$

- Lagrangian $L = T - U$

$$= \tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) + Ar^{\alpha-1}$$

- Lagrange's equations

for r: $\frac{\partial L}{\partial r} - \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = 0$

$$mr\dot{\phi}^2 + (\alpha-1)Ar^{\alpha-2} - \frac{d}{dt}(m\dot{r}) = 0$$

$$m\ddot{r} = A(\alpha-1)r^{\alpha-2} + mr\dot{\phi}^2$$

for ϕ: $\cancel{\frac{\partial L}{\partial \phi}} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} = 0$

0

$$\frac{d}{dt}(mr^2\dot{\phi}) = 0$$

Recall: $l = m\vec{r}\times\vec{v} = mrv_\perp = mr(r\dot{\phi}) = mr^2\dot{\phi}$

so this tells us $\frac{dl}{dt} = 0$, i.e. angular momentum is conserved ✓

- Is energy conserved?

Following pp. 261–262: Since the Lagrangian is not an explicit function of time, H is conserved. Since the polar coords are not related to cartesians by an explicit time dependence, H is the energy of the system. So, E is conserved.

Could also write $E = T + U$ and differentiate directly

$$\frac{d}{dt}(T+U) \stackrel{?}{=} 0$$

If we write $T = \tfrac{1}{2}m\dot{r}^2 + \tfrac{1}{2}l\dot{\phi}$, and use $\frac{dl}{dt} = 0$

we see $\frac{dT}{dt} = m\dot{r}\ddot{r} + \tfrac{1}{2}l\ddot{\phi}$

$$\frac{dU}{dt} = \frac{\partial U}{\partial r}\dot{r} = -A(\alpha-1)r^{\alpha-2}\dot{r}$$

so $\frac{dE}{dt} = [m\ddot{r} - A(\alpha-1)r^{\alpha-2}]\dot{r} + \tfrac{1}{2}l\ddot{\phi}$

using r eqn: $= +mr\dot{r}\dot{\phi}^2 + \tfrac{1}{2}l\ddot{\phi}$

$$= +mr\dot{r}\dot{\phi}^2 + \tfrac{1}{2}mr^2\dot{\phi}\ddot{\phi}$$
$$= \tfrac{1}{2}mr\dot{\phi}[+2\dot{r}\dot{\phi} + r\ddot{\phi}]$$
$$= \tfrac{1}{2}\dot{\phi}\cdot\frac{dl}{dt} = 0$$ ✓