from example 7.8 :

Constraints: $y_1 + y_2 = l_2$, $h = y_1 + y_2 - l_2 = 0$

$x_1 + x_2 = l_1$, $f = x_1 + x_2 - l_1 = 0$

$$T_1 = \tfrac{1}{2} m_1 \dot{x}_1^2$$

$$T_2 = \tfrac{1}{2} m_2 (\dot{x}_2 + \dot{y}_1)^2$$

$$T_3 = \tfrac{1}{2} m_3 (\dot{x}_2 + \dot{y}_2)^2$$

$$U_1 = -m_1 g x_1$$

$$U_2 = -m_2 g (x_2 + y_1)$$

$$U_3 = -m_3 g (x_2 + y_2)$$

- $L = T - U = \tfrac{1}{2} m_1 \dot{x}_1^2 + \tfrac{1}{2} m_2 (\dot{x}_2 + \dot{y}_1)^2 + \tfrac{1}{2} m_3 (\dot{x}_2 + \dot{y}_2)^2 +$
$$m_1 g x_1 + m_2 g (x_2 + y_1) + m_3 g (x_2 + y_2)$$

- Lagrange's eqns with constraints are as per eq. 7.65 :

① $\dfrac{\partial L}{\partial x_1} - \dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{x}_1} + \lambda_1 \underbrace{\dfrac{\partial}{\partial x_1}(x_1 + x_2 - l_1)}_{f} + \lambda_2 \underbrace{\dfrac{\partial}{\partial x_2}(y_1 + y_2 - l_2)}_{h} = 0$

$$m_1 g - \dfrac{d}{dt}(m_1 \dot{x}_1) + \lambda_1 + 0 = 0$$

$$m_1 g - m_1 \ddot{x}_1 + \lambda_1 = 0$$

$$m_1 \ddot{x}_1 = m_1 g + \lambda_1$$

② $\dfrac{\partial L}{\partial x_2} - \dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{x}_2} + \lambda_1 \dfrac{\partial}{\partial x_2} f + \lambda_2 \dfrac{\partial}{\partial x_1} h = 0$

$$(m_2 + m_3) g - \dfrac{d}{dt}\left[m_2(\dot{x}_2 + \dot{y}_1) + m_3(\dot{x}_2 + \dot{y}_2) \right] + \lambda_1 = 0$$

$$(m_3 + m_2)\ddot{x}_2 + m_2 \ddot{y}_1 + m_3 \ddot{y}_2 = (m_2 + m_3) g + \lambda_1$$

③ $\frac{\partial L}{\partial y_1} - \frac{d}{dt}\frac{\partial L}{\partial \dot{y}_1} + \lambda_1 \frac{\partial f}{\partial y_1} + \lambda_2 \frac{\partial h}{\partial y_1} = 0$

$$m_2 g - \frac{d}{dt}\left[m_2(\ddot{x}_2 + \dot{y}_1)\right] + \lambda_1 \cdot 0 + \lambda_2 = 0$$

$$m_2(\ddot{x}_2 + \ddot{y}_1) = (m_2 + m_3)g + \lambda_2$$

④ $\frac{\partial L}{\partial y_2} - \frac{d}{dt}\frac{\partial L}{\partial \dot{y}_2} + \lambda_1 \frac{\partial f}{\partial y_2} + \lambda_2 \frac{\partial h}{\partial y_2} = 0$

$$m_3 g - \frac{d}{dt}\left[m_3(\dot{x}_2 + \dot{y}_2)\right] + 0 + \lambda_2 = 0$$

$$m_3(\ddot{x}_2 + \ddot{y}_2) = m_3 g + \lambda_2$$

Collect :

① $\quad m_1 \ddot{x}_1 = m_1 g + \lambda_1$

② $\quad (m_2 + m_3)\ddot{x}_2 + m_2 \ddot{y}_1 + m_3 \ddot{y}_2 = (m_2 + m_3)g + \lambda_1$

③ $\quad m_2(\ddot{x}_2 + \ddot{y}_1) = m_2 g + \lambda_2$

④ $\quad m_3(\ddot{x}_2 + \ddot{y}_2) = m_3 g + \lambda_2$

and the constraints : $\quad y_1 + y_2 - \ell_2 = 0 \implies \ddot{y}_2 = -\ddot{y}_1$

$\qquad\qquad\qquad\qquad x_1 + x_2 - \ell_1 = 0 \implies \ddot{x}_2 = -\ddot{x}_1$

Solve : $\qquad$ First use constraint eqns. to eliminate x_2, y_2

① $\quad m_1 \ddot{x}_1 = m_1 g + \lambda_1$

② $\quad -(m_2 + m_3)\ddot{x}_1 + (m_2 - m_3)\ddot{y}_1 = (m_2 + m_3)g + \lambda_1$

③ $\quad -m_2(\ddot{x}_1 - \ddot{y}_1) = m_2 g + \lambda_2$

④ $\quad -m_3(\ddot{x}_1 + \ddot{y}_1) = m_3 g + \lambda_2$

∘ $\;$ Observe : ② $-$ (③ $+$④) $\implies \quad \lambda_1 = 2\lambda_2 = 0$

$\qquad\qquad\qquad\qquad\qquad\qquad \lambda_1 = 2\lambda_2$

This 'uses up' eq. ②...

Remaining eqns:

① $\quad m_1 \ddot{x}_1 = m_1 g + \lambda_1 \qquad\qquad \longrightarrow \quad \ddot{x}_1 = g + \frac{\lambda_1}{m_1}$

③ $\quad -m_2(\ddot{x}_1 - \ddot{y}_1) = m_2 g + \frac{\lambda_1}{2} \qquad \rightsquigarrow \quad \ddot{x}_1 - \ddot{y}_1 = -g - \frac{\lambda_1}{2m_2}$

④ $\quad -m_3(\ddot{x}_1 + \ddot{y}_1) = m_3 g + \frac{\lambda_1}{2} \qquad \rightsquigarrow \quad \ddot{x}_1 + \ddot{y}_1 = -g - \frac{\lambda_1}{2m_3}$

③ $+$④ $\longrightarrow \quad 2\ddot{x}_1 = -2g - \frac{\lambda_1}{2}\left(\frac{1}{m_2} + \frac{1}{m_3}\right)$

⑤ $\qquad\qquad \ddot{x}_1 = -g - \frac{\lambda_1}{4}\left(\frac{1}{m_2} + \frac{1}{m_3}\right)$

comparing with ① : $\qquad -\frac{\lambda_1}{4}\left(\frac{1}{m_2} + \frac{1}{m_3}\right) - g = g + \frac{\lambda_1}{m_1}$

$$\lambda_1\left[\frac{1}{m_1} + \frac{1}{4}\left(\frac{1}{m_2} + \frac{1}{m_3}\right)\right] = -2g$$

$$\lambda_1 = \frac{-2g}{\frac{1}{m_1} + \frac{1}{4}\left(\frac{1}{m_2} + \frac{1}{m_3}\right)} \quad , \quad \lambda_2 = \frac{\lambda_1}{2}$$
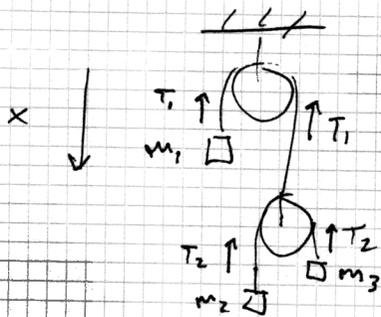
Take m_1 as system :

① $\quad \Sigma F = m_1 \ddot{x}_1$

$\quad m_1 g - T_1 = m_1 \ddot{x}_1$
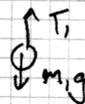
② $\quad \Sigma F = m_2 (\ddot{x}_2 + \ddot{y}_1)$

$\quad m_2 g - T_2 = m_2 (\ddot{x}_2 + \ddot{y}_1)$ T_2 with m_2 as system
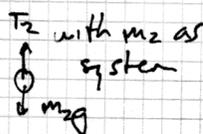
③ $\quad \Sigma F = m_3 (\ddot{x}_2 + \ddot{y}_2)$

$\quad m_3 g - T_2 = m_3 (\ddot{x}_2 + \ddot{y}_2)$ with m_3 as system.

In these three equations, T_2 and T_1 play the same roles as $-\lambda_2$, $-\lambda_1$.
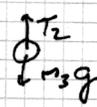
A final equation can be obtained by taking the massless lower pulley as system.

There we see $\Sigma F = ma = 0$

$$-2T_2 + T_1 = 0, \text{ which is}$$

analogous to

$$\lambda_1 = 2\lambda_2 \quad \text{obtained from the fourth independ. eqn. in our earlier calculation.}$$
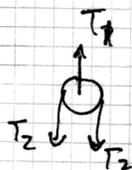
So: λ_1 and λ_2 really are the (negatives of the) string tensions.