TM 7-25

helix $\begin{cases} z = k\theta & \longleftrightarrow \theta = \frac{z}{k} \\ r = \text{const} \end{cases}$

choose cylindrical coords (r, θ, z)

$$T = \tfrac{1}{2} m v^2$$

$$= \tfrac{1}{2} m\left((r\dot\theta)^2 + \cancel{\dot r^2}^{\,0} + \dot z^2\right) = \tfrac{1}{2} m\left(\left(\tfrac{r}{k}\right)^2 \dot z^2 + \dot z^2\right) = \tfrac{1}{2} m\left(\left(\tfrac{r}{k}\right)^2 + 1\right)\dot z^2$$

$$U = mgz$$

- $L = T - U = \tfrac{1}{2} m\left(\left(\tfrac{r}{k}\right)^2 + 1\right)\dot z^2 - U$
- corresponding momentum: $p = \frac{\partial L}{\partial \dot z} = m\left(\left(\tfrac{r}{k}\right)^2 + 1\right)\dot z \Rightarrow \dot z = \frac{p}{m\left(\left(\frac{r}{k}\right)^2 + 1\right)}$
- Hamiltonian: $H = p\dot z - L$

$$= p\dot z - \tfrac{1}{2} m\left(\left(\tfrac{r}{k}\right)^2 + 1\right)\dot z^2 + mgz$$

$$= \frac{p^2}{m\left(\left(\frac{r}{k}\right)^2 + 1\right)} - \frac{1}{2}\frac{p^2}{m\left(\left(\frac{r}{k}\right)^2 + 1\right)} + mgz$$

$$H = \frac{p^2}{2m\left(\left(\frac{r}{k}\right)^2 + 1\right)} + mgz$$

Hamiltonian eqns:

$$\boxed{\begin{aligned} \frac{\partial H}{\partial p} = \dot z &\Longrightarrow \frac{p}{m\left(\left(\frac{r}{k}\right)^2 + 1\right)} = \dot z \\ -\frac{\partial H}{\partial z} = \dot p &\Longrightarrow \dot p = -mg \end{aligned}}$$

so: eqn of motion is, putting these together,

$$\ddot z\, m\left(\left(\tfrac{r}{k}\right)^2 + 1\right) = -mg$$

$$\ddot z = \frac{-g}{\left(\left(\frac{r}{k}\right)^2 + 1\right)}$$