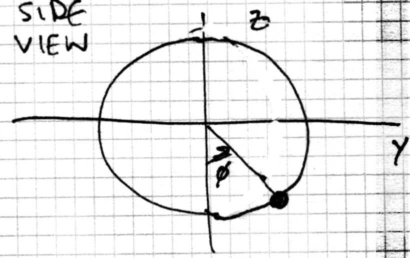
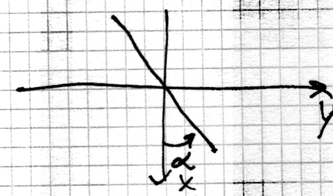
TM 7-21

SIDE VIEW



TOP VIEW



$$\frac{d\alpha}{dt} = \omega = \text{const.}$$

- Calculate T, U and L:

$$x = (R\sin\phi)\cos\alpha \quad\longrightarrow\quad \dot{x} = (R\cos\phi\,\dot{\phi})\cos\alpha + (R\sin\phi)(-\sin\alpha\,\dot{\alpha})$$
$$= R[\cos\phi\cos\alpha\,\dot{\phi} - \sin\phi\sin\alpha\,\omega]$$
$$y = (R\sin\phi)\sin\alpha \quad\longrightarrow\quad \dot{y} = (R\cos\phi\,\dot{\phi})\sin\alpha + (R\sin\phi)(\cos\alpha\,\dot{\alpha})$$
$$= R[\cos\phi\sin\alpha\,\dot{\phi} + \sin\phi\cos\alpha\,\omega]$$
$$z = -R\cos\phi \quad\longrightarrow\quad \dot{z} = R\sin\phi\,\dot{\phi}$$

$$v^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2$$
$$= R^2[\cos\phi\cos\alpha\,\dot{\phi} - \sin\phi\sin\alpha\,\omega]^2 + R^2[\cos\phi\sin\alpha\,\dot{\phi} + \sin\phi\cos\alpha\,\omega]^2 + R^2\sin^2\phi\,\dot{\phi}^2$$
$$= R^2\left[\dot{\phi}^2(\cos^2\phi(\cos^2\alpha + \sin^2\alpha) + \sin^2\phi) + \omega^2(\sin^2\phi(\sin^2\alpha + \cos^2\alpha))\right.$$
$$\left. + 2\dot{\phi}\omega[-\cos\phi\cos\alpha\sin\phi\sin\alpha + \cos\phi\cos\alpha\sin\phi\sin\alpha]\right]$$
$$= R^2[\dot{\phi}^2 + \omega^2\sin^2\phi]$$

$$T = \tfrac{1}{2}mv^2 = \tfrac{1}{2}mR^2[\dot{\phi}^2 + \omega^2\sin^2\phi]$$
$$U = mgz = -mgR\cos\phi$$
$$L = T - U = \tfrac{1}{2}mR^2[\dot{\phi}^2 + \omega^2\sin^2\phi] + mgR\cos\phi$$

- Lagrange's equation:

$$\frac{\partial L}{\partial \phi} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} = 0$$

$$\frac{\partial L}{\partial \phi} = mR^2\omega^2\sin\phi\cos\phi - mgR\sin\phi$$

$$\frac{\partial L}{\partial \dot{\phi}} = mR^2\dot{\phi}$$

$$-mR^2\ddot{\phi} + mR^2\omega^2\sin\phi\cos\phi - mgR\sin\phi = 0$$
$$\ddot{\phi} + \sin\phi\left(-\omega^2\cos\phi + \frac{g}{R}\right) = 0$$

Equilibrium points? soln's to eqns. when $\ddot{\phi} = \dot{\phi} = 0$ $\quad(0 \le \phi \le \pi)$

i.e. $\sin\phi\left(\frac{g}{R} - \omega^2\cos\phi\right) = 0$

$\sin\phi = 0$ or $\omega^2\cos\phi = \frac{g}{R}$

$\phi = 0, \pi$ $\qquad \cos\phi = \frac{g}{\omega^2 R}$

Note that $\cos\phi = \frac{g}{\omega^2 R}$ has a solution only when $0 \le \frac{g}{\omega^2 R} \le 1$.

i.e. $\frac{g}{R} \le \omega^2$. So, $\omega_c^2 = \frac{g}{R}$ is a critical value for ω. Below this value there are two equilibria, while for ω above ω_c there are three equilibria.

Which are stable? Check by considering small perturbations (linear) about the equilibrium point. If the perturbations grow the equilibrium is unstable. If they don't it's stable.

① about $\phi_{eq} = 0$: $\phi = \phi_{eq} + \delta = \delta$

$$\begin{cases} \sin\phi \sim \delta \\ \cos\phi \sim 1 \end{cases}$$

equation of motion in this approx is:

$$\ddot{\delta} + \delta\left(\frac{g}{R} - \omega^2\right) = 0$$

$$\ddot{\delta} + \delta(\omega_c^2 - \omega^2) = 0$$

- For $\omega < \omega_c$ this is a harmonic oscillator eqn. Solution is of form

  $\delta = \delta_0 \cos\Omega t$, where $\Omega = \sqrt{\omega_c^2 - \omega^2}$

  i.e. oscillation has constant amplitude — doesn't grow

- for $\omega > \omega_c$, the solution is an expontial

  $\delta = Ae^{-\sqrt{\omega^2-\omega_c^2}\,t} + Be^{\sqrt{\omega^2-\omega_c^2}\,t}$

  which will grow for $t \to \infty$ due to second term.
  This is UNSTABLE.

- $\omega = \omega_c$: $\ddot{\delta} = 0 \rightarrow \delta = At + B$.
  Perturbation grows.. unstable

② about $\phi_{eq} = \pi$: $\phi = \phi_{eq} + \delta = \pi + \delta$

Taylor exp: $\sin\phi = \sin\phi_{eq} + \delta\cos\phi_{eq} + \dots = -\delta$

$\cos\phi \approx -1$

Eqn. of motion: $\ddot{\delta} - \delta\left(\frac{g}{R} + \omega^2\right) = 0$

$$\ddot{\delta} - \delta(\omega_c^2 + \omega^2) = 0$$

This always has exponential sol'n, which grows

$$\delta = Ae^{\sqrt{\omega_c^2+\omega^2}\,t} + Be^{-\sqrt{\omega_c^2+\omega^2}\,t}$$

<u>UNSTABLE</u>

3 $\phi_{eq} = \cos^{-1}\left(\frac{g}{\omega^2 R}\right)$; $\phi = \phi_{eq} + \delta$

Taylor exp: $\begin{cases} \sin\phi = \sin\phi_{eq} + \delta\cos\phi_{eq} + O(\delta^2) \\ \cos\phi = \cos\phi_{eq} - \delta\sin\phi_{eq} + O(\delta^2) \end{cases}$

Eqn. of motion:

$$\ddot{\delta} + (\sin\phi_{eq} + \delta\cos\phi_{eq})\left[\frac{g}{R} - \omega^2(\cos\phi_{eq} - \delta\sin\phi_{eq})\right] = 0$$

: $$\ddot{\delta} + (\sin\phi_{eq} + \delta\cos\phi_{eq})\,\omega^2\sin\phi_{eq}\,\delta = 0$$

to $O(\delta)$ only: $$\ddot{\delta} + (\omega^2\sin^2\phi_{eq})\,\delta = 0$$

$$\cos^2\phi_{eq} = \left(\frac{g}{\omega^2 R}\right)^2 = \left(\frac{\omega_c}{\omega}\right)^4$$

$$\sin^2\phi_{eq} = 1 - \cos^2\phi_{eq} = 1 - \left(\frac{\omega_c}{\omega}\right)^4$$

$$\omega^2\sin^2\phi_{eq} = \frac{\omega^4 - \omega_c^4}{\omega^2} > 0 \quad (\text{for } \omega > \omega_c)$$

This has an oscillatory sol'n

$$\delta = A\cos(\Omega t + \phi_0), \quad \Omega = \sqrt{\frac{\omega^4 - \omega_c^4}{\omega^2}}$$

so is a STABLE equilibrium.

So: For $\omega < \omega_c$: $\phi = 0$ is stable, $\phi = \pi$ is unstable

$\omega > \omega_c$: $\phi = 0$ is unstable, $\phi = \cos^{-1}\left(\frac{\omega_c}{\omega}\right)^2$ is stable, $\phi = \pi$ unstable.

Physically: below ω_c, the gravitational force dominates the centrifugal force. For $\omega > \omega_c$, the component of the centrifugal force along the ring can match the component of gravity tangent to the ring. The angle at which this balancing occurs is the equilibrium angle, ϕ_{eq}. As ω gets larger and the centrifugal force is stronger, the angle of equilibrium gets larger (particle swings out farther)