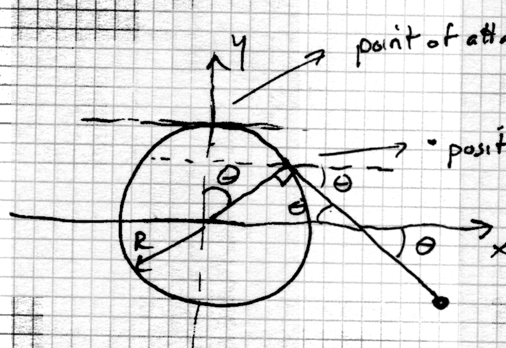

- position of contact: $(x_1, y_1) = (R\sin\theta, R\cos\theta)$
- length of string wrapped on circle is $R\theta$
- length of remaining string (forming free pendulum): $l - R\theta = d$

- position of mass:

$$(x_2, y_2) = (x_1 + d\cos\theta,\ y_1 - d\sin\theta)$$
$$= (R\sin\theta + (l-R\theta)\cos\theta,$$
$$R\cos\theta - (l-R\theta)\sin\theta)$$

$$\dot{x}_2 = R\cos\theta\,\dot\theta - R\cos\theta\,\dot\theta - (l-R\theta)\dot\theta\sin\theta = -\dot\theta\sin\theta\,(l-R\theta)$$
$$\dot{y}_2 = -R\sin\theta\,\dot\theta + R\dot\theta\sin\theta - (l-R\theta)\cos\theta\,\dot\theta = -\dot\theta\cos\theta\,(l-R\theta)$$

$$T = \frac{1}{2}m(\dot{x}_2^2 + \dot{y}_2^2) = \frac{1}{2}m\dot\theta^2(l-R\theta)^2(\sin^2\theta + \cos^2\theta) = \frac{1}{2}m\dot\theta^2(l-R\theta)^2$$

$$U = mgy_2 = mg[R\cos\theta - (l-R\theta)\sin\theta]$$

$$L = T - U = \frac{1}{2}m\dot\theta^2(l-R\theta)^2 - mg\,R\cos\theta + (l-R\theta)\sin\theta\ mg$$

<u>Equation of motion</u>:

$$\frac{\partial L}{\partial\theta} = \frac{1}{2}m\dot\theta^2\,2(l-R\theta)(-R) + mgR\sin\theta + mg(-R)\sin\theta$$
$$+ mg(l-R\theta)\cos\theta$$

$$= -m\dot\theta^2R(l-R\theta) + mg(l-R\theta)\cos\theta$$
$$= m(l-R\theta)[-\dot\theta^2R + g\cos\theta]$$

$$\frac{\partial L}{\partial\dot\theta} = m\dot\theta\,(l-R\theta)^2$$

$$\frac{d}{dt}\frac{\partial L}{\partial\dot\theta} = m\ddot\theta\,(l-R\theta)^2 + m\dot\theta\,2(l-R\theta)(-R\dot\theta)$$
$$= m\ddot\theta\,(l-R\theta)^2 - 2mR(l-R\theta)\dot\theta^2$$

$$= m(l-R\theta)[\ddot\theta(l-R\theta) - 2R\dot\theta^2]$$

so: $m(l-R\theta)[\ddot\theta(l-R\theta) - 2R\dot\theta^2] = m(l-R\theta)[-\dot\theta^2R + g\cos\theta]$

$$\ddot\theta(l-R\theta) - 2R\dot\theta^2 = -\dot\theta^2R + g\cos\theta$$

$$\boxed{\ddot\theta(l-R\theta) - R\dot\theta^2 - g\cos\theta = 0}$$

- <u>Equilibrium point</u>? $\ddot\theta = \dot\theta = 0 \longrightarrow g\cos\theta = 0$

$\theta = \frac{\pi}{2}$ (hanging straight down)

- Frequency of small oscillations? (about equilibrium pt)
  (expand in a series about equilibrium point)
  define: $\theta = \frac{\pi}{2} + \phi$
  Eqn. of motion becomes

$$\ddot\phi\left(l - R\left(\phi + \tfrac{\pi}{2}\right)\right) - R\dot\phi^2 - g\cos\left(\tfrac{\pi}{2} + \phi\right) = 0$$

$$\cos\tfrac{\pi}{2}\cos\phi - \sin\tfrac{\pi}{2}\sin\phi = -\sin\phi \qquad \left(\cos\tfrac{\pi}{2} = 0\right)$$

$$\ddot\phi\left(\left(l - R\tfrac{\pi}{2}\right) - R\phi\right) - R\dot\phi^2 + g\sin\phi = 0$$

- For ϕ very small, $\sin\phi \sim \phi$
- Keeping only terms up to $O(\phi)$ in the eqn. of motion:
  (so dropping $\ddot\phi\phi$ and $\dot\phi^2$ as $O(\phi^2)$)

$$\ddot\phi\left(l - R\tfrac{\pi}{2}\right) + g\phi = 0$$

$$\ddot\phi + \frac{g}{l - R\frac{\pi}{2}}\phi = 0$$

since $l - R\frac{\pi}{2} > 0$, this is a harmonic oscillator eqn.

$$\ddot\phi + \omega_0^2\phi = 0$$

where $\omega_0^2 = \frac{g}{l - R\frac{\pi}{2}}$

$$\boxed{\omega_0 = \sqrt{\frac{g}{l - R\frac{\pi}{2}}}}$$

- Line about which angular motion extends equally in opposite directions?

If we considered, more generally than above, small oscillations about some arbitrary angle θ_0,

so $\theta = \theta_0 + \phi$, $\cos\theta \approx \cos\theta_0 - \phi\sin\theta_0$

the eqn. of motion is:

$$\ddot\phi\left(l - R(\theta_0 + \phi)\right) - R\dot\phi^2 - g\cos(\theta_0 + \phi) = 0$$

to 1st order in ϕ

$$\ddot\phi\left(l - R\theta_0\right) - g(\cos\theta_0 - \phi\sin\theta_0) = 0$$

$$\ddot\phi\left(l - R\theta_0\right) + g\sin\theta_0\,\phi = g\cos\theta_0$$

$$\ddot\phi + \frac{g}{l - R\theta_0}\sin\theta_0\,\phi = \frac{g\cos\theta_0}{l - R\theta_0} \qquad \Big| \quad \omega_0^2 = \frac{g\sin\theta_0}{l - R\theta_0}$$

This is a SHO with a constant 'force', so has sol'n

$$\phi = \phi_{homog} + \phi_{particular} = A\cos(\omega_0 t + \delta) + \frac{\cos\theta_0}{\sin\theta_0}$$

when the $\frac{\cos\theta_0}{\sin\theta_0}$ term vanishes, this corresponds to angular osc. from $\phi = A$ to $\phi = -A$ about θ_0. <u>This occurs when $\theta_0 = \frac{\pi}{2}$</u>.