11-3. The equation of an ellipsoid is

$$\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} + \frac{x_3^2}{c^2} = 1 \tag{1}$$

which can be written in normalized form if we make the following substitutions:

$$x_1 = a\xi, \quad x_2 = b\eta, \quad x_3 = c\zeta \tag{2}$$

Then, Eq. (1) reduces to

$$\xi^2 + \eta^2 + \zeta^2 = 1 \tag{3}$$

This is the equation of a sphere in the (ξ, η, ζ) system.

If we denote by dv the volume element in the x_i system and by $d\tau$ the volume element in the (ξ, η, ζ) system, we notice that the volume of the ellipsoid is

$$V = \int dv = \int dx_1 \, dx_2 \, dx_3 = abc \int d\xi \, d\eta \, d\zeta$$

$$= abc \int d\tau = \frac{4}{3} \pi abc \tag{4}$$

because $\int d\tau$ is just the volume of a sphere of unit radius.

The rotational inertia with respect to the x_3-axis passing through the center of mass of the ellipsoid (we assume the ellipsoid to be homogeneous), is given by

$$I_3 = \frac{M}{V} \int \left(x_1^2 + x_2^2 \right) dv$$

$$= \frac{M}{V} abc \int \left(a^2 \xi^2 + b^2 \eta^2 \right) d\tau \tag{5}$$

In order to evaluate this integral, consider the following equivalent integral in which $z = r \cos \theta$:

$$\int a^2 z^2 \, dv = \int a^2 z^2 \left(r \, dr \, r \sin \theta \, d\theta \, d\phi \right)$$

$$= a^2 \int_0^{2\pi} d\phi \int_0^{\pi} \cos^2 \theta \sin \theta \, d\theta \int_0^{R=1} r^4 \, dr$$

$$= a^2 \times 2\pi \times \frac{2}{3} \times \frac{1}{5}$$

$$= \frac{4\pi a^2}{15} \tag{6}$$

Therefore,

$$\int \left(a^2 \xi^2 + b^2 \eta^2 \right) d\tau = \frac{4\pi}{15} \left(a^2 + b^2 \right) \tag{7}$$

and

$$\boxed{I_3 = \frac{1}{5} M \left(a^2 + b^2 \right)} \tag{8}$$

Since the same analysis can be applied for any axis, the other moments of inertia are

$$\boxed{\begin{aligned} I_1 &= \frac{1}{5} M\left(b^2 + c^2\right) \\ I_2 &= \frac{1}{5} M\left(a^2 + c^2\right) \end{aligned}}$$

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