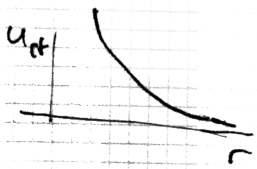
8.14 ★★★ Consider a particle of reduced mass μ orbiting in a central force with $U = kr^n$ where $kn > 0$. **(a)** Explain what the condition $kn > 0$ tells us about the force. Sketch the effective potential energy U_{eff} for the cases that $n = 2, -1$, and -3. **(b)** Find the radius at which the particle (with given angular momentum ℓ) can orbit at a fixed radius. For what values of n is this circular orbit stable? Do your sketches confirm this conclusion? **(c)** For the stable case, show that the period of small oscillations about the circular orbit is $\tau_{\text{osc}} = \tau_{\text{orb}}/\sqrt{n+2}$. Argue that if $\sqrt{n+2}$ is a rational number, these orbits are closed. Sketch them for the cases that $n = 2, -1$, and 7.

$U = kr^n$, $kn > 0$

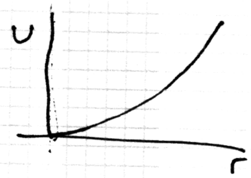
(a) what does kn indicate about force?

$F = -\frac{\partial U}{\partial r} = -knr^{n-1} \Rightarrow$ $kn > 0 \Rightarrow F(r) < 0$, i.e. force is in $-\hat{r}$ direction, so is attractive.

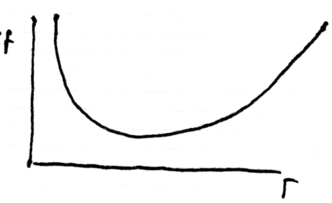
$U_{cf} = \frac{\ell_z^2}{2\mu r^2}$



(i) for $n = 2$ $\quad U = kr^2$

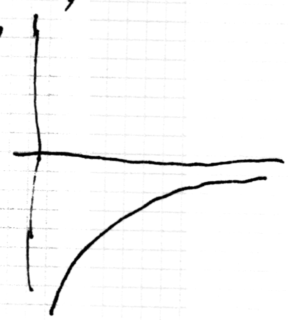


$U_{eff} = U_{cf} + U \Rightarrow$

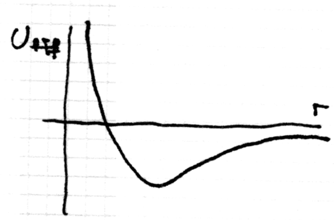


(ii) for $-1 = n$, with $k < 0$,

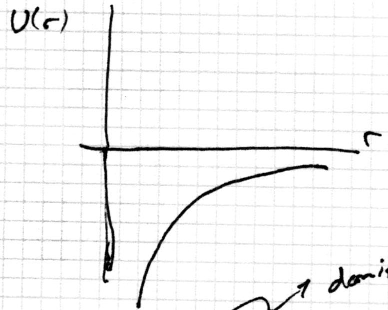
$U = \frac{-|k|}{r}$



$U_{eff} = U_{cf} + U$

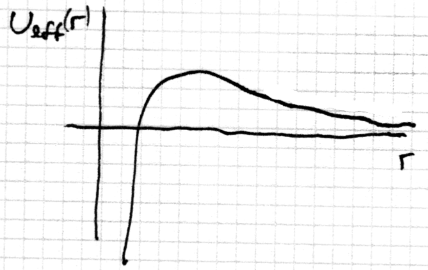


(iii) $n=-3,\ k<0$



dominates for smaller r

$$U_{eff} = -\frac{|k|}{r^3} + \frac{l_z^2}{2\mu r^2}$$ dominates for larger r



(b) $$U_{eff} = U(r) + \frac{l_z^2}{2\mu r^2} = kr^n + \frac{l_z^2}{2\mu r^2}$$

radial eqn: $m\ddot{r} = -\frac{\partial U_{eff}}{\partial r}$

→ want values of r for which $r=\text{const.}$ For these values of r, will have $\ddot{r}=0$, so want sol'ns to

$$\left.\frac{\partial U_{eff}}{\partial r}\right|_{r=r_0} = 0$$

$$\left.\frac{\partial U_{eff}}{\partial r}\right|_{r=r_0} = knr_0^{n-1} - \frac{l_z^2}{\mu r_0^3} = 0$$

$$knr_0^{n-1} = \frac{l_z^2}{kn\mu r_0^3}$$

$$r_0^{n+2} = \frac{l_z^2}{kn\mu}$$

$$r_0 = \left[\frac{l_z^2}{kn\mu}\right]^{\frac{1}{n+2}} \quad \text{(assume } kn>0\text{)}$$

→ For what values of n is this stable?

Orbit is stable if $\left.\frac{\partial^2 U}{\partial r^2}\right|_{r=r_0} > 0$.

$$\frac{\partial^2 U_{eff}}{\partial r^2} = \frac{\partial}{\partial r}\left[knr^{n-1} - \frac{l_z^2}{\mu r^3}\right] = kn(n-1)r^{n-2} + \frac{3l_z^2}{\mu r^4}$$

at $r = r_0$,

$$\left.\frac{d^2 U_{eff}}{dr^2}\right|_{r=r_0} = k\,n(n-1)\,r_0^{n-2} + \frac{3 l_z^2}{\mu r_0^4} > 0$$

multiply by r_0^4:

$$k n(n-1)\, r_0^{n+2} + \frac{3 l_z^2}{\mu} > 0$$

$$\cancel{kn}(n-1)\,\frac{l_z^2}{\cancel{kn}\mu} + \frac{3 l_z^2}{\mu} > 0$$

$$(n-1) + 3 > 0$$

$$n + 2 > 0$$

$$n > -2 \quad \text{for stable orbits.}$$

This is consistent w/ sketches:

- $n = 2, -1$ have minima of U at $r = r_0$
- for $n = -3$ it's a local max at $r = r_0$.

ⓒ Show that period of small oscillations about circular orbit is:

$$\tau_{osc} = \frac{\tau_{orb}}{\sqrt{n+2}}$$

- radial equation: $\mu \ddot{r} = -\dfrac{\partial U_{eff}(r)}{\partial r}$

For small perturbations about r_0

$$r = r_0 + \epsilon$$

- $\mu \ddot{r} \to \mu \ddot{\epsilon}$
- $\dfrac{\partial U_{eff}}{\partial r} \approx \underbrace{\cancel{\dfrac{\partial U_{eff}(r_0)}{\partial r}}}_{\to 0} + \epsilon\,\dfrac{\partial^2 U_{eff}(r_0)}{\partial r^2}$

eqn of motion is: $\mu \ddot{\epsilon} + \epsilon\,\dfrac{\partial^2 U_{eff}(r_0)}{\partial r^2} = 0$

- $\dfrac{\partial^2 U_{eff}(r_0)}{\partial r^2} = k n(n-1)\, r_0^{n-2} + \dfrac{3 l_z^2}{\mu r_0^4} = \dfrac{1}{r_0^4}\left(kn(n-1)\, r_0^{n+2} + \dfrac{3 l_z^2}{\mu}\right)$

$$= \frac{1}{r_0^4}\left[\cancel{kn}(n-1)\,\frac{l_z^2}{\cancel{kn}\mu} + \frac{3 l_z^2}{\mu}\right] = \frac{l_z^2}{\mu}\,\frac{1}{r_0^4}\,(n+2)$$

so

$$\ddot{\epsilon} + \frac{\epsilon}{\mu}\,\frac{\partial^2 U_{eff}(r_0)}{\partial r^2} = 0$$

$$\ddot{\epsilon} + \omega_0^2 \epsilon = 0, \qquad \omega_0^2 = \frac{1}{\mu}\,\frac{\partial^2 U_{eff}(r_0)}{\partial r^2} = \frac{l_z^2}{\mu^2}\,\frac{n+2}{r_0^4}$$

and $\tau_{rad} = \dfrac{2\pi}{\omega_0} = 2\pi\,\dfrac{\mu}{l_z}\,\dfrac{r_0^2}{\sqrt{n+2}}$

Period of angular motion?

$$\dot{\phi} = \frac{l_z}{\mu r_0^2} \implies \frac{\Delta\phi}{\Delta t} = \frac{l_z}{\mu r_0^2}. \text{ For } \Delta\phi = 2\pi,$$

we get $\Delta t = \tau$.

So: $\tau_{ang} = \dfrac{2\pi \mu r_0^2}{l_z}$

using $r_0 = \left[\dfrac{l_z^2}{kn\mu}\right]^{\frac{1}{n+2}}$,

we get

$$\tau_{ang} = \frac{2\pi\mu r_0^2}{l_z}$$

so: $\tau_{rad} = \dfrac{\tau_{ang}}{\sqrt{n+2}}$ $\left(\text{i.e. } \tau_{osc} = \dfrac{\tau_{orb}}{\sqrt{n+2}}\right)$

- If $\sqrt{n+2}$ is rational, call it $\frac{p}{q}$, we have

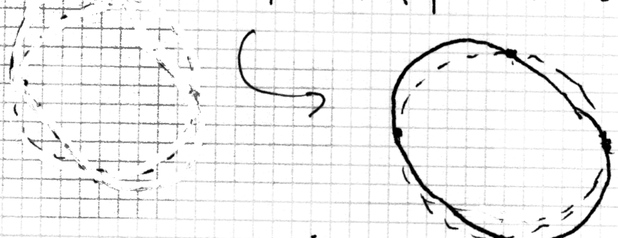
  $\tau_{rad} = \frac{q}{p}\tau_{ang}$. After a time $p\tau_{rad} = q\tau_{ang} = T$, the angular position and the radial position are the same as the initial positions and the orbit closes.

- $n=2$: $\tau_{osc} = \dfrac{\tau_{orb}}{2} \implies \tau_{orb} = 2\tau_{osc}$

  $n=-1$: $\tau_{osc} = \tau_{orb}$.

  $n=7$: $\tau_{osc} = \dfrac{\tau_{orb}}{3} \implies \tau_{orb} = 3\tau_{osc}$.

$\underline{n=2}$: 2 oscillations per orbit (so oscillating path crosses circular orbit 4 times per orbit)



$n=-1$: 1 osc. per orbit (osc. path crosses circular orbit twice per orbit)

$n=7$: 3 osc. per orbit (osc. path crosses circ. orbit 6 times per orbit)