7.29 ★★ Figure 7.14 shows a simple pendulum (mass m, length l) whose point of support P is attached to the edge of a wheel (center O, radius R) that is forced to rotate at a fixed angular velocity ω. At $t = 0$, the point P is level with O on the right. Write down the Lagrangian and find the equation of motion for the angle ϕ. [*Hint:* Be careful writing down the kinetic energy T. A safe way to get the velocity right is to write down the position of the bob at time t, and then differentiate.] Check that your answer makes sense in the special case that $\omega = 0$.

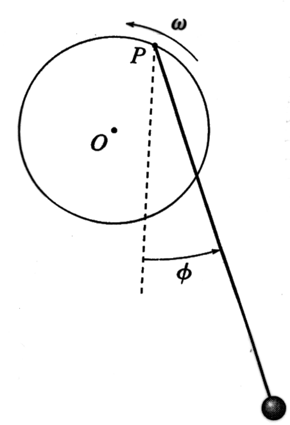
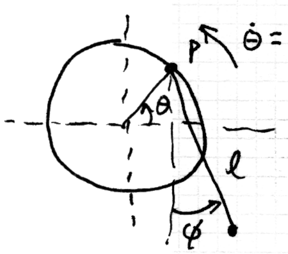


Figure 7.14 Problem 7.29

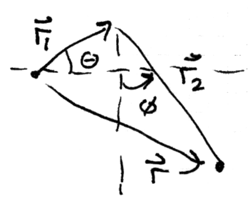


$L = T - U$

$T = \frac{1}{2} m |\dot{\vec{r}}|^2$

$\vec{r} = \vec{r}_1 + \vec{r}_2$

$\theta = \omega t$



$$\vec{r}_1 = (\hat{x}\cos\theta + \hat{y}\sin\theta) R$$

$$\vec{r}_2 = (\hat{x}\sin\phi + \hat{y}(-\cos\phi))$$

$$\vec{r} = \vec{r}_1 + \vec{r}_2 = \hat{x}[R\cos\theta + l\sin\phi] + \hat{y}[R\sin\theta - l\cos\phi]$$

$$\vec{r} = \hat{x}[R\cos\omega t + l\sin\phi] + \hat{y}[R\sin\omega t - l\cos\phi]$$

$$\dot{\vec{r}} = \hat{x}[-R\omega\sin\omega t + l\cos\phi\,\dot{\phi}] + \hat{y}[R\omega\cos\omega t + l\sin\phi\,\dot{\phi}]$$

$$|\dot{\vec{r}}|^2 = [-R\omega\sin\omega t + l\cos\phi\,\dot{\phi}]^2 + [R\omega\cos\omega t + l\sin\phi\,\dot{\phi}]^2$$

$$= (R\omega)^2\sin^2\omega t + l^2\cos^2\phi\,\dot{\phi}^2 - 2R\omega l\sin\omega t\cos\phi\,\dot{\phi} + (R\omega)^2\cos^2\omega t + l^2\sin^2\phi\,\dot{\phi}^2 + 2R\omega l\cos\omega t\sin\phi\,\dot{\phi}$$

$$= (R\omega)^2 + l^2\dot{\phi}^2 + 2Rl\omega\dot{\phi}[\cos\omega t\sin\phi - \sin\omega t\cos\phi]$$

$$= (R\omega)^2 + l^2\dot{\phi}^2 + 2Rl\omega\dot{\phi}\sin(\phi - \omega t)$$

$$T = \frac{1}{2}m[(R\omega)^2 + \ell^2\dot\phi^2 + 2R\ell\omega\dot\phi\sin(\phi-\omega t)]$$

$$U = mgy = mg[\ -\ell\cos\phi\ + R\sin\omega t]$$

$$L = \frac{1}{2}m[(R\omega)^2 + \ell^2\dot\phi^2 + 2R\ell\omega\dot\phi\sin(\phi-\omega t)] - mg[\ -\ell\cos\phi + R\sin\omega t]$$

Lagrange eq: $\frac{\partial L}{\partial \phi} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot\phi}\right) = 0$

$$\frac{\partial L}{\partial \phi} = \frac{1}{2}m\ 2R\ell\omega\dot\phi\cos(\phi-\omega t) - mg\ell\sin\phi$$
$$= m\ell[R\omega\dot\phi\cos(\phi-\omega t) - g\sin\phi]$$

$$\frac{\partial L}{\partial \dot\phi} = \frac{1}{2}m[2\ell^2\dot\phi + 2R\ell\omega\sin(\phi-\omega t)] = \ell m[\ell\dot\phi + R\omega\sin(\phi-\omega t)]$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\phi}\right) = \ell m[\ell\ddot\phi + R\omega\cos(\phi-\omega t)\ (\dot\phi-\omega)]$$

$$0 = \frac{\partial L}{\partial \phi} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot\phi}\right) = m\ell[R\omega\dot\phi\cos(\phi-\omega t) - g\sin\phi]$$
$$- m\ell[\ell\ddot\phi + R\omega(\dot\phi-\omega)\cos(\phi-\omega t)]$$

$$0 = -g\sin\phi - \ell\ddot\phi + R\omega^2\cos(\phi-\omega t)$$

$$\boxed{\ddot\phi = -\frac{g}{\ell}\sin\phi + \frac{R}{\ell}\omega^2\cos(\phi-\omega t)}$$

for $\omega = 0$, $\ddot\phi = -\frac{g}{\ell}\sin\phi$, which is exactly the equation of motion for a pendulum.