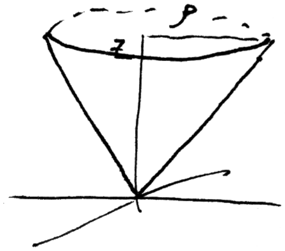
6.17 ★★ Find the geodesics on the cone whose equation in cylindrical polar coordinates is $z = \lambda\rho$. [Let the required curve have the form $\phi = \phi(\rho)$.] Check your result for the case that $\lambda \to 0$.

Geodesics on a cone? (let curve have form $\phi = \phi(\rho)$)

$z = \lambda\rho$



In cylindrical coords,
- a small move in z-direction add dist. (Δz)
- a small move in ρ-dir. adds dist $\Delta\rho$
- a small move in ϕ-dir. adds dist $\rho\Delta\phi$

So:

$$\Delta s = \sqrt{(\Delta z)^2 + (\Delta\rho)^2 + (\rho\Delta\phi)^2}$$

in cylindrical coords.

using $z = \lambda\rho \Rightarrow \Delta z = \lambda\Delta\rho$, we have

$$\Delta s = \sqrt{\lambda^2(\Delta\rho)^2 + (\Delta\rho)^2 + (\rho\Delta\phi)^2}$$

$$= \sqrt{(\lambda^2+1)(\Delta\rho)^2 + (\rho\Delta\phi)^2}$$

$$= \Delta\rho\sqrt{(\lambda^2+1) + (\rho\frac{\Delta\phi}{\Delta\rho})^2}$$

$$\Delta s = \Delta\rho\sqrt{(\lambda^2+1) + (\rho\frac{d\phi(\rho)}{d\rho})^2}$$

- A geodesic should extremize distance, i.e.

$$L = \int_{\rho_1}^{\rho_2}\Delta s = \int_{\rho_1}^{\rho_2} d\rho\,[(\lambda^2+1) + (\rho\frac{d\phi}{d\rho})^2]^{1/2}$$

should be stationary for geodesic.

We can apply the Euler-Lagrange equations:

$$\frac{\partial f}{\partial\phi} - \frac{d}{d\rho}\frac{\partial f}{\partial\phi'} = 0$$

where $f[\phi, \phi'; \rho] = [(\lambda^2+1) + \rho^2(\phi')^2]^{1/2}$

observe: $\frac{\partial f}{\partial\phi} = 0$, so

$$\frac{d}{d\rho}\left[\frac{\partial f}{\partial\phi'}\right] = 0 \Rightarrow \frac{\partial f}{\partial\phi'} = C$$ some constant.

$$c = \frac{\partial f}{\partial \phi'} = \frac{\partial}{\partial \phi'}\left[(\lambda^2+1) + \rho^2(\phi')^2\right]^{1/2}$$

$$= \frac{1}{2}\frac{1}{\left[(\lambda^2+1)+\rho^2(\phi')^2\right]^{1/2}}\rho^2 \, 2\phi'$$

$$c = \frac{\rho^2\phi'}{\left[(\lambda^2+1)+\rho^2(\phi')^2\right]^{1/2}}$$

$$c^2\left[(\lambda^2+1)+\rho^2(\phi')^2\right] = \rho^4(\phi')^2$$

$$(\phi')^2\left[\rho^4 - c^2\rho^2\right] = c^2(\lambda^2+1)$$

$$(\phi')^2\rho^2\left[\rho^2-c^2\right] = c^2(\lambda^2+1)$$

$$(\phi')^2 = \frac{c^2(\lambda^2+1)}{\rho^2\left[\rho^2 - c^2\right]}$$

$$\phi' = \frac{c\sqrt{\lambda^2+1}}{\rho\sqrt{\rho^2-c^2}}$$

$$\phi(\rho) = \phi_0 + \sqrt{\lambda^2+1}\int d\rho \, \frac{c}{\rho\sqrt{\rho^2-c^2}}$$

Integration constant

$\rho = cx \qquad \leftrightarrow x = \frac{\rho}{c}$

$d\rho = c\,dx$

$$= \phi_0 + \sqrt{\lambda^2+1}\underbrace{\int dx\,\frac{1}{x\sqrt{x^2-1}}}$$

this is on the inner cover of Taylor's book: $= \cos^{-1}(\frac{1}{x})$

$$\phi(\rho) = \phi_0 + \sqrt{\lambda^2+1}\cos^{-1}\left(\frac{1}{x}\right)$$

$$\boxed{\phi(\rho) = \phi_0 + \sqrt{\lambda^2+1}\cos^{-1}\left(\frac{c}{\rho}\right)}$$

- For $\lambda \to 0$, i.e. the cone approaches the x-y plane ($z \to 0$)

$$\phi(\rho) = \phi_0 + \cos^{-1}\left(\frac{c}{\rho}\right)$$

$$\cos(\phi(\rho)-\phi_0) = \frac{c}{\rho}$$

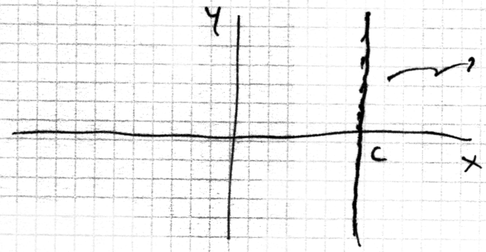
$$\rho = \frac{c}{\cos(\phi(\rho)-\phi_0)}$$

$$\rho\cos(\phi(\rho)-\phi_0) = c$$

What does this look like?

If $\phi_0 = 0$, i.e. $\rho\cos\phi = c$, we'd recognize it as $x = c$

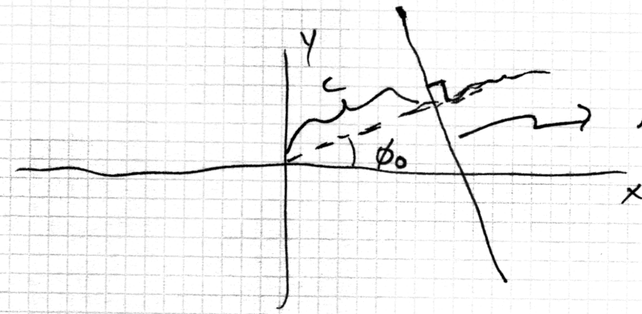
i.e.



$\rho \cos\phi = c$;

a straight line perpendicular to $\phi = 0$, passing a minimum distance c from the origin.

For $\phi_0 \neq 0$, this is a straight line $\perp$ to the $\phi = \phi_0$ line, passing a minimum distance c from the origin.



$\rho \cos(\phi - \phi_0) = c$