Complex Numbers and AC Circuits

In these notes we will explore the application of complex numbers to the description of currents and voltages in AC circuits. We will at first limit ourselves to the case of simple periodic behavior (sines and cosines). Later we will see how to generalize these results to other types of time dependence.

I. A Capacitor in an AC Circuit.

Consider the simple series AC circuit:

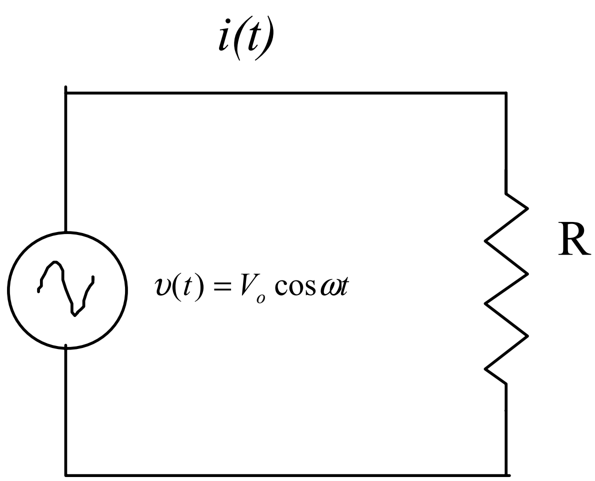


Figure 1.

We know that $i(t) = \upsilon(t)/R = (V/R)\cos\omega t = I\cos\omega t$, since R is an Ohmic device (I is directly proportional to V). Further, if we add a second resistor, R', to the circuit, either in series or in parallel with R, it is relatively easy to calculate how the current being drawn from the voltage source will be affected and how the voltage drop across R will change (if at all).

N.B. We will follow the convention of using lower case letters like $i(t)$ for time-dependent quantities and upper case letters like V, the amplitude of the voltage, for time-independent quantities. We need to be careful about not confusing i when it refers to current and $i = \sqrt{-1}$. Electrical engineers avoid this difficulty by using $j = \sqrt{-1}$

On the other hand, if we add a capacitor to the above circuit, it is much more difficult to calculate how voltages and currents are altered. This difficulty arises because for the capacitor (unlike the resistor), i(t) is not a simple linear function of υ(t). If υ_C is the voltage drop across the capacitor, then

$$i(t) = C\frac{d\upsilon_C}{dt} \quad . \tag{1}$$

The fact that i(t) depends not upon υ_C but upon its derivative introduces a phase shift between current and voltage in circuits containing capacitors. For example, in the circuit

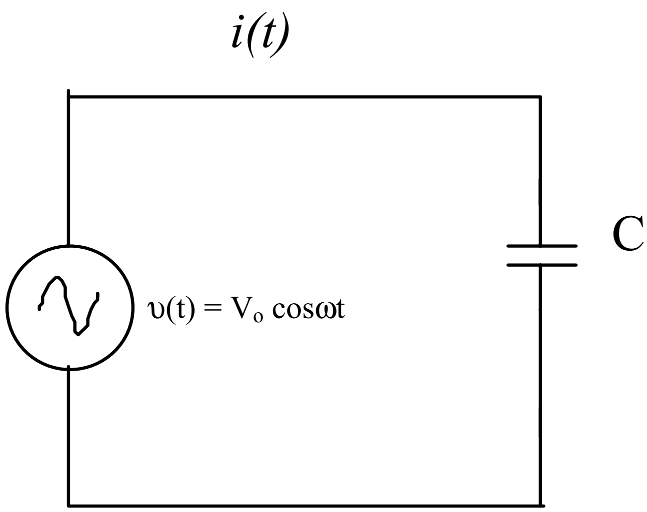


Figure 2

shown in Fig. 2, assume that the voltage across the capacitor is given by $\upsilon_C(t) = V\cos\omega t$, then the current is expressed by

$$i(t) = C\ du_C / dt = -\ C\omega\ V\sin(\omega t) \tag{2}$$

or

$$i(t) = C\omega V\cos(\omega t + \pi/2) \tag{3}$$

Eq. (3) tells us that the current leads the voltage across the capacitor by 1/4 of a cycle or 90^{o}. Can you give a physical explanation of this phase difference?

II. Complex Phasors

By using the algebra of complex numbers, we can simplify the analysis of circuits containing capacitors, inductors, and resistors. The simplification results from the fact that $e^{i\omega t}$ has simpler mathematical properties than either cos ωt or sin ωt. In particular, the derivative of $e^{i\omega t}$ is proportional to $e^{i\omega t}$, whereas a similar proportionality doesn't hold for sin ωt or cosωt.

Our procedure for using complex numbers will be as follows: Let us first introduce complex representations of voltages and currents. For example, a voltage can always be expressed by

$$v(t) = V_0 \cos(\omega t + \phi) = \frac{1}{2} V_0 \left[e^{i(\omega t + \phi)} + e^{-i(\omega t + \phi)} \right] \equiv \frac{1}{2} \left[V e^{i\omega t} + V^* e^{-i\omega t} \right]. \tag{4}$$

V is called the *phasor* representing the voltage $\upsilon(t)$. In general, a phasor will be a complex number that can be written as an amplitude multiplied by a phase factor:

$$V = |V| e^{i\phi} \tag{5}$$

Note that the amplitude of the phasor is the same as the amplitude V_o. We also note that the last term in Eq. (4) is just the complex conjugate of the penultimate term. Thus, if we know one, we easily find the other. Hence in our circuit analysis, we will simply replace $\upsilon(t)$ with its phasor representation $Ve^{i\omega t}$. The current is represented by a similar expression

$$i(t) = Ie^{i\omega t}. \tag{6}$$

[These replacements can be justified more rigorously by substituting Eq. (4) and its equivalent for $i(t)$ in the differential equation that describes the circuit behavior.]

A comment:

If you insist on writing an expression for $\upsilon(t)$ or $i(t)$, which as we shall see is completely unnecessary in most cases, these expressions can be obtained by taking the real part of the phasor expression: $\upsilon(t) = Re[Ve^{i\omega t}]$.

Now let's apply these notions to the circuit shown in Figure 3.

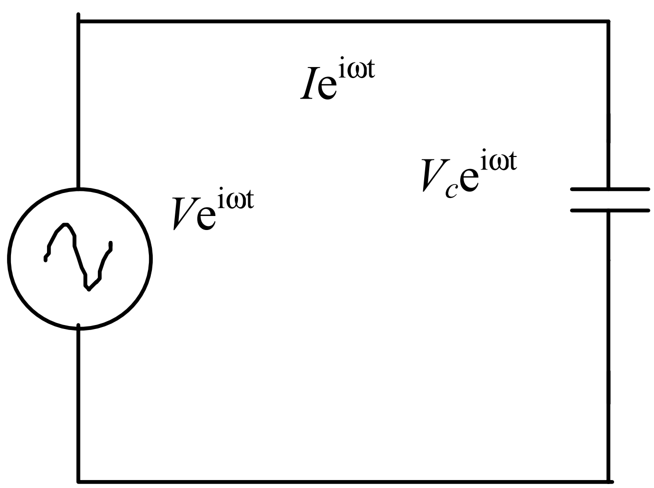


Figure 3

We can relate the current to the voltage across the capacitor, $i(t) = C\, d\upsilon_c/dt$. Replacing $i(t)$ and $\upsilon_c(t)$ with their phasor representations yields

$$I = Ci\omega V_C \tag{7}$$

which defines the **capacitive impedance**

$$Z_C = -i/\omega C\,. \tag{8}$$

We see that in terms of complex currents and voltage, I is proportional to V just as for the simple resistor case. So, for simply periodic currents and voltages, we can write a **Generalized Ohm's Law**:

$$V = IZ\,, \tag{9}$$

where in general I, V, and Z are complex numbers.

We can use the phasors to read off the phase difference between $i(t)$ and $\upsilon(t)$ as well as the amplitude of $i(t)$. We rewrite the phasor as

$$I = |I|e^{i\phi} = \omega C e^{i\pi/2} V\,. \tag{10}$$

Eq. (10) tells us that the amplitude of the current is ωC times the amplitude of the voltage and that the current leads the voltage by $\pi/2$ or 90°. (Recall that multiplying a complex number by $e^{i\theta}$ is equivalent to rotating the phasor counter-clockwise by the angle θ on the Argand diagram.)

If we want to return to the explicit time dependence for i(t), which we never really need, we can write

$$i(t) = \omega C V_0 \cos\left(\omega t + \frac{\pi}{2}\right) \tag{11}$$

III. A More Complex Example.

To see more fully the power of using complex numbers to analyze AC circuits, consider the circuit in Figure 4.

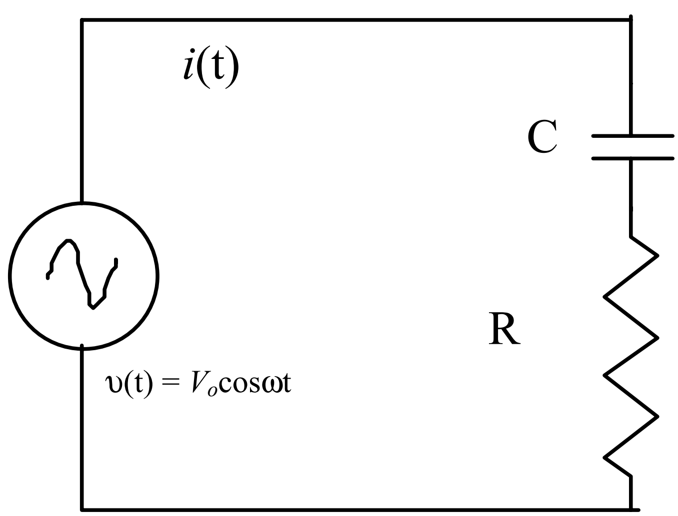
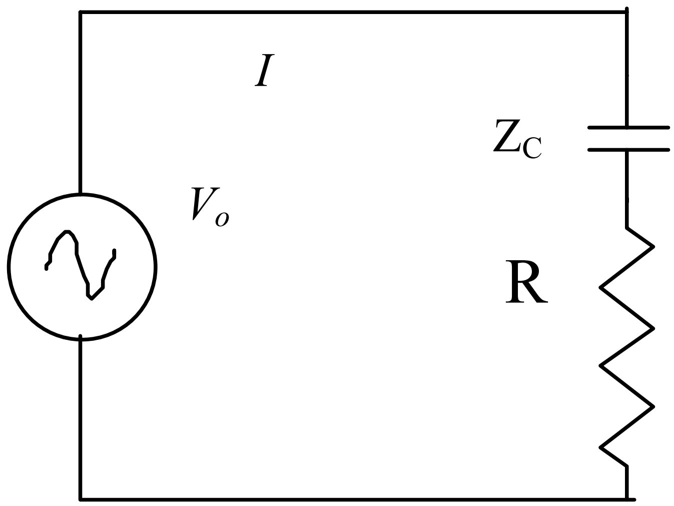


Figure 4

We'll replace the symbols in this circuit by the phasors and impedances as shown in Fig.



5.

Figure 5

In Fig. 5, we have used $Z_C = -i/\omega C$. We then combine the resistance and capacitive impedance using the sum rule for impedances in series to write

$$I = \frac{V_0}{R + Z_C} = \frac{V_0}{R - \frac{i}{\omega C}} = \frac{V_0}{R^2 + \frac{1}{\omega^2 C^2}}\left(R + \frac{i}{\omega C}\right). \tag{12}$$

We rewrite the last factor as

$$R + \frac{i}{\omega C} = \sqrt{R^2 + \frac{1}{\omega^2 C^2}}\, e^{i\phi}, \quad \text{where } \phi = tan^{-1}(\, 1/R\omega C\,). \tag{13}$$

Therefore, we find for the current phasor

$$I = \frac{V_o e^{i\phi}}{\sqrt{R^2 + \frac{1}{\omega^2 C^2}}}. \tag{14}$$

We read from Eq. (14) that the current leads the voltage $\upsilon(t)$ by the frequency dependent phase angle ϕ.

Now the (complex) voltage drop across R is

$$V_R = IR = \frac{R V_o e^{i\phi}}{\sqrt{R^2 + \frac{1}{\omega^2 C^2}}} \tag{15}$$

and the (complex) voltage drop across C is

$$V_C = Z_C I = \frac{-\frac{i}{\omega C} V_o e^{i\phi}}{\sqrt{R^2 + 1/\omega^2 C^2}} = \frac{V_o e^{i(\phi - \pi/2)}}{\sqrt{(\omega RC)^2 + 1}} . \qquad (16)1$$

IV. Inductors and Inductive Impedance

Consider the AC circuit with an inductor shown in Figure 6.

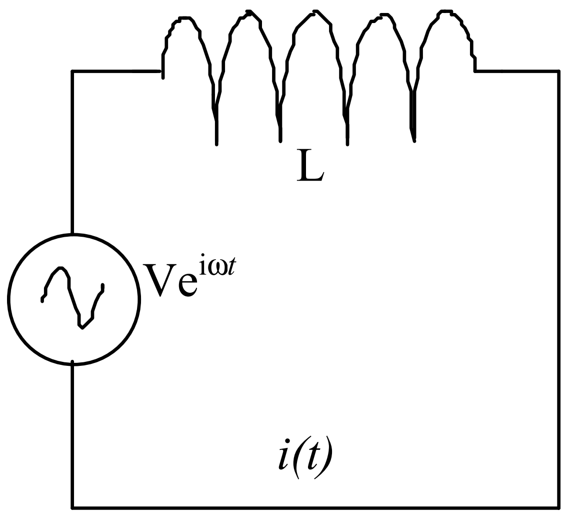


Figure 6

We know that an inductor produces an emf proportional to the rate of change of the current passing through it: $L di/dt = v(t)$. Using the phasor form for the current we see that

$$I = V/(i\omega L) = V/Z_L , \qquad (17)$$

which defines the inductive impedance $Z_L = i\omega L$.

V. Series LRC Circuit

We now combine our results to analyze the series LRC circuit shown in Figure 7.

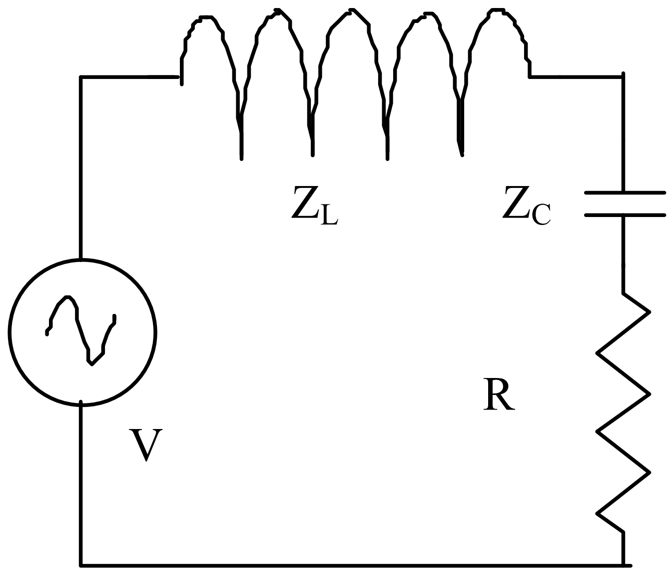


Figure 7

The combined impedances add for a series circuit, so we may immediately write

$$I = \frac{V}{R + Z_C + Z_L} \tag{18}$$

or

$$I = \frac{V}{R + i(\omega L - 1/\omega C)} \tag{19}$$

Putting Eq. (19) into amplitude-and-phase form yields

$$I = \frac{V_0 e^{i\phi}}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} \text{ and } \phi = \tan^{-1}\left(-\frac{\omega L - \frac{1}{\omega C}}{R}\right) \tag{20}$$

Note: $\phi < 0$ if $\omega L > 1/\omega C$ and $\phi > 0$ if $\omega L < 1/\omega C$. The "resonance condition" (largest amplitude for the current) occurs when $\omega^2 = 1/LC$, $\phi = 0$, and $I = V/R = I_{max}$.

VI. Power and Phasors: A Cautionary Tale

Suppose the voltage drop across a device is $V_o \cos \omega t$ and the current through the device is $I_o \cos(\omega t + \phi)$. The time-dependent (instantaneous) power dissipated in the device is

$$p(t) = i(t)\upsilon(t) = I_o V_o \cos\omega t \ \cos(\omega t + \phi)\,. \tag{21}$$

This result implies that if we use phasors to represent voltages and currents, we must say that

$$p(t) = \mathrm{Re}\left(I\, e^{i\omega t}\right) \times \mathrm{Re}\left(V\, e^{i\omega t}\right) = \mathrm{Re}\left(|I|\, e^{i(\omega t+\phi)}\right) \times \mathrm{Re}\left(|V|\, e^{i\omega t}\right) \tag{22}$$

and not $P = \mathrm{Re}\{ I\ V \}$ to get the **time-dependent** power.

However, in many cases, we are concerned with the time-averaged power:

$$\overline{P} = \frac{1}{T}\int_0^{\frac{2\pi}{\omega}} p(t)dt = \frac{1}{2} I_0 V_0 \cos\phi \tag{23}$$

or

$$\overline{P} = I_{RMS} V_{RMS} \cos\phi \tag{24}$$

where

$$I_{RMS} = \frac{I_o}{\sqrt{2}} \quad \text{and} \quad V_{RMS} = \frac{V_o}{\sqrt{2}} \tag{25}$$

If we make use of the phasor notation, the time-averaged power is easily found:

$$\overline{P} = \frac{1}{2}\mathrm{Re}\left[I\, V^*\right] = \frac{1}{2}\mathrm{Re}\left[I^* V\right] \tag{26}$$

Note: If the device is either a capacitor or an inductor (with zero resistance), then

$$\phi = \pm\frac{\pi}{2} \text{ , and } \overline{P} = 0 \text{ .}$$

That is, there is no power dissipation, on average, in a purely capacitive or purely inductive device.

In general, for devices that consist of a collection of resistors, capacitors, and inductors, the (complex) impedance of the device can always be written $Z_{\text{eff}} = R_{\text{eff}} + i X_{\text{eff}}$ where R_{eff} is the resistive part of Z_{eff} and X_{eff} is the reactive part of Z_{eff}. Using phasors, we write

$$I = \frac{V_0}{2\left|Z_{eff}\right|} e^{i\phi}, \phi = \tan^{-1}\left(\frac{-X_{eff}}{R_{eff}}\right) \tag{28}$$

and the time average power is

$$\overline{P} = \frac{V_o^2}{2\,|\,Z_{eff}\,|}\cos\phi = \frac{R_{eff}\, V_o^2}{2\,|\,Z_{eff}\,|^2} = \frac{R_{eff}\, V_o^2}{2[\, R_{eff}^2 + X_{eff}^2\,]} \tag{29}$$

Note that if $R_{eff} = 0$, then $\overline{P} = 0$.